Group Theory

Groningen, 2nd year bachelor mathematics, 2018

J. Top, J.S. Müller

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preface

These lecture notes contain a translation into English of the Dutch lecture notes on Group Theory as they were used in the mathematics curriculum of Groningen University during the period 1993–2013, with some modifications added later. The original Dutch text may be found at `http://www.math.rug.nl/~top/alg1.pdf`.

Both the present text and the original are loosely based on another Dutch text on Group Theory, called *Algebra I*, written in the late 1970's at the university of Amsterdam by Prof.dr. F. Oort and Prof.dr. H.W. Lenstra.

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Jaap Top, Steffen Müller

I THE INTEGERS

In this chapter we consider the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$. We learn many elementary properties of these in elementary and secondary education. However, in most cases formal proofs of these properties are not discussed there. Such proofs form the main part of the present chapter, and may be viewed as a repetition and extension of the same subject as it was treated during part of the first year bachelor's course 'Kaleidoscope Mathematics'. In the second chapter of the present notes we will see how the developed theory about integers is used, for example, in order to obtain a better understanding of computations involving remainders upon division by a fixed integer. In turn, calculations using such remainders will be used to obtain criteria determining whether a given large integer is or is not a prime number.

Here as well as in subsequent chapters, many examples will be found illustrating how rather abstract definitions and proofs turn out to be quite applicable in concrete situations.

I.1 Division (with remainder)

In elementary school one encounters exercises like $100 : 7 = 14\,\mathrm{R}\,2$, meaning that 7 goes into 100 in total 14 times, leaving a remainder of 2. The following general fact is behind problems of this sort.

I.1.1 Theorem. (Division with remainder.) *Let $a, b \in \mathbb{Z}$ with $b \neq 0$. Then there exist $q, r \in \mathbb{Z}$ such that*

$$a = qb + r \quad \textit{and} \quad 0 \leq r < |b|.$$

Moreover, these q, r are unique.

Proof. We first show existence of the required $q, r \in \mathbb{Z}$ Let us assume $a \geq 0$. We prove existence using mathematical induction with respect to a: in case $a = 0$ one may take $q = r = 0$. Now let $a > 0$ and use as induction hypothesis that $a - 1 \geq 0$ can be written as $a - 1 = \tilde{q}b + \tilde{r}$ with $0 \leq \tilde{r} < |b|$. Then $a = \tilde{q}b + \tilde{r} + 1$. Here evidently $0 \leq \tilde{r} + 1 \leq |b|$. In case $\tilde{r} + 1 < |b|$ we may take $q = \tilde{q}$ and $r = \tilde{r} + 1$. In the remaining case $\tilde{r} + 1 = |b|$ we have

$$a = \tilde{q}b + \tilde{r} + 1 = \tilde{q}b + |b| = (\tilde{q} + \frac{|b|}{b})b + 0.$$

Hence one can take $q = \tilde{q} + \frac{|b|}{b}$ and $r = 0$. Using the principle of mathematical induction this proves existence of q, r in the case $a \geq 0$.

If $a < 0$, then $-a > 0$, hence the argument above shows that there exist q', r' with $-a = q'b + r'$ and $0 \le r' < |b|$. Then $a = (-q')b - r' = (-q' - \frac{|b|}{b})b + (|b| - r')$, so we conclude that $q = -q'$ and $r = 0$ work in case $r' = 0$, while for $r' \neq 0$ one can take $q = -q' - \frac{|b|}{b}$ and $r = |b| - r'$.

It remains to prove uniqueness. Suppose that $a = q_1 b + r_1 = q_2 b + r_2$ with $0 \le r_1 \le r_2 < |b|$. Then $0 \le r_2 - r_1 \le r_2 < |b|$, and also $r_2 - r_1 = b(q_1 - q_2)$. Hence $r_1 = r_2$, since otherwise $r_2 - r_1$ would be a positive multiple of $|b|$, contradicting $r_2 - r_1 < |b|$. As a consequence $b(q_1 - q_2) = 0$, and since $b \neq 0$ this implies $q_1 = q_2$. This proves the theorem. ∎

I.1.2 Remark. With a little knowledge about real numbers, a different argument may be given: partition the real line in intervals of length $|b|$, so

$$\mathbb{R} = \ldots \cup [-2|b|, -|b|) \cup [-|b|, 0) \cup [0, |b|) \cup \ldots\ldots$$

Then $a \in \mathbb{R}$ is in exactly one such interval, hence can be written as $a = qb + r$ with $0 \le r < |b|$.

A reason to prefer the proof using induction over the argument involving real numbers, is that conceptually $\mathbb{R}$ is much more difficult than $\mathbb{Z}$. In fact, one can *construct* $\mathbb{R}$ by first constructing the rational numbers $\mathbb{Q}$ starting from $\mathbb{Z}$, and then building $\mathbb{R}$ from $\mathbb{Q}$ via a technique called 'completion'.

I.1.3 Definition. Let $a, b \in \mathbb{Z}$. We say that a *divides* the integer b, if $q \in \mathbb{Z}$ exists such that $b = qa$. This is denoted by $a|b$. In case no such q exists, we write $a \nmid b$, and we say that a does *not* divide b.

Instead of a divides b one also says that a is a *divisor* of b, or that a is a *factor* of b, or that b is a *multiple* of a, or that b is *divisible* by a. For example $17|-153$ and $0|0$ and $-2 \nmid 101$ and $0 \nmid 3$.

We present some elementary properties of divisibility.

I.1.4 Proposition. *For $a, b, c \in \mathbb{Z}$ one has:*

1. *If $a|b$ and $b|c$, then also $a|c$.*
2. *If $a|b$ and $a|c$, then also $a|b \pm c$.*
3. *$a|0$ and $1|a$.*
4. *$0|a$ if and only if $a = 0$.*
5. *If $b \neq 0$ and $a|b$, then $|a| \le |b|$.*

Proof. These properties are immediate consequences of the definition. As an example, $a|b$ and $a|c$ implies that $p, q \in \mathbb{Z}$ exist with $b = pa$ and $c = qa$, and hence it follows that $b \pm c = pa \pm qa = (p \pm q)a$, so $a|b \pm c$. You should try to find formal proofs of the other properties. ∎

A consequence of the last of the properties mentioned in Proposition I.1.4, is that an integer $a \neq 0$ has only finitely many divisors; the largest of these is evidently $|a|$. In particular, if b is another integer, then a and b have only finitely many divisors in common (two of the common divisors are of course 1 and -1). If one considers common multiples of two integers a, b, then in case a or b equals 0, the fourth property mentioned in Proposition I.1.4 implies that 0 is the only common multiple. However, in case $ab \neq 0$ the integers a and b have *positive* common multiples, for example $|ab|$. Hence the following definition makes sense:

I.1.5 Definition. Let $a, b \in \mathbb{Z}$. In case a and b are not both equal to 0, the *greatest common divisor* of a and b is defined as the largest integer that is a divisor of both a and b. This integer is denoted as $\gcd(a, b)$. Furthermore, we define $\gcd(0, 0) := 0$.

We define the *least common multiple* of a and b, denoted by $\mathrm{lcm}(a,b)$, to be 0 if $ab = 0$, and to be the smallest positive integer k satisfying both $a|k$ and $b|k$ otherwise.

Two integers a, b are call *coprime* or *relative prime* if $\gcd(a,b) = 1$.

I.1.6 Example. We have $\gcd(a,b) = \gcd(b,a)$ and $\gcd(a,0) = |a|$. Since a and $|a|$ have the same divisors, it follows that $\gcd(a,b) = \gcd(|a|,b) = \gcd(a,|b|) = \gcd(|a|,|b|)$. It is a difficult task in general to compute a greatest common divisor directly from the definition. Try, for example, to check that $\gcd(35581, 46189) = 221$.

One easily constructs similar examples with the lcm; for instance, one encounters them in school when one tries to find a common denominator for two fractions. In the remainder of this section we will only discuss the gcd; in Section I.2 we will return to the notion lcm. ▬

It turns out that a surprisingly simple and efficient algorithm exists for computing $\gcd(a,b)$. This dates back from the Greek mathematician Euclid who lived around 300 B.C. The algorithm runs as follows.

I.1.7 Theorem. (The Euclidean algorithm.) *The following algorithm computes the greatest common divisor of two integers a, b in finitely many steps:*

```
gcd:=proc(a::integer,b::integer)::integer;
        local rn,ro,help;
        ro:=max(abs(a),abs(b)); rn:=min(abs(a),abs(b));
        while rn<>0 do
            do help:=ro; ro:=rn; rn:=help mod rn end do;
        return ro
    end proc;
```

Proof. To understand this program, we check what happens during the 'while-loop'. Each time this loop is executed, the pair of integers (ro, rn) is replaced by (rn, ro mod rn). Here ro mod rn is the remainder upon dividing ro by rn. In particular, at the start of the 'loop' it holds that ro, rn ≥ 0, and each time the loop is executed, rn becomes strictly smaller. Hence the program terminates.

To show that it indeed computes the greatest common divisor of a and b, we show that, moreover, when entering the 'loop', we always have $\gcd(\mathrm{ro},\mathrm{rn}) = \gcd(a,b)$. This will be done in Lemma I.1.9 below. Accepting the lemma, it follows that after the last execution of the 'loop' $\mathrm{rn} = 0$ and $\gcd(a,b) = \gcd(\mathrm{ro},\mathrm{rn}) = \gcd(\mathrm{ro},0) = \mathrm{ro}$. In other words, indeed the algorithm outputs the greatest common divisor of a and b. ■

I.1.8 Remark. In many programming languages, for *negative* a the result of a mod b is *not* the remainder r as given in Theorem I.1.1, but it is $r - |b|$. The code above is written in Maple; here this problem does not occur.

I.1.9 Lemma. *For $a, b, q, r \in \mathbb{Z}$ with $a = qb + r$ we have* $\gcd(a,b) = \gcd(b,r)$.

Proof. We will show that the set of common divisors of a and b equals the set of common divisors of b and r. The definition of greatest common divisor then implies the lemma.

If $d|a$ and $d|b$, then also $d|a - qb = r$. Hence common divisors of a and b are also common divisors of b and r.

Vice versa, if $d|b$ and $d|r$, then also $d|qb + r = a$. Hence the common divisors of b and r are also common divisors of a and b. This proves the lemma. ■

I.1.10 Example.

$$\begin{aligned}
\gcd(1057,315) &= \gcd(3\cdot 315+112,315) \\
&= \gcd(315,112) && = \gcd(2\cdot 112+91,112) \\
&= \gcd(112,91) && = \gcd(91+21,91) \\
&= \gcd(91,21) && = \gcd(4\cdot 21+7,21) \\
&= \gcd(21,7) && = \gcd(7,0) = 7.
\end{aligned}$$

■

We now discuss the efficiency of the Euclidean algorithm.

I.1.11 Theorem. (G. Lamé, 1844, French mathematician.) *If $a > b > 0$, then the number of divisions with remainder performed by the Euclidean algorithm when determining $\gcd(a,b)$ is at most 5 times the number of decimal digits of b.*

Proof. Write $r_0 = a$ and $r_1 = b$. The algorithm computes one by one

$$\begin{array}{lcll}
r_0 & = & q_0 r_1 + r_2 & (0 < r_2 < r_1) \\
r_1 & = & q_1 r_2 + r_3 & (0 < r_3 < r_2) \\
r_2 & = & q_2 r_3 + r_4 & (0 < r_4 < r_3) \\
 & & \vdots & \\
r_{n-2} & = & q_{n-2} r_{n-1} + r_n & (0 < r_n < r_{n-1}) \\
r_{n-1} & = & q_{n-1} r_n + 0. &
\end{array}$$

The number of divisions with remainder is therefore exactly n.

To estimate n we use the so-called Fibonacci sequence $(f_i)_{i\geq 0}$, defined inductively by $f_0 = f_1 = 1$ and $f_{i+2} = f_{i+1} + f_i$ for $i \geq 0$. So this is the sequence $1,1,2,3,5,8,13,21,34,\ldots\ldots$.

Using mathematical induction, we now show $r_{n-i} \geq f_{i+1}$ for $i = 0,\ldots,n-1$. The case $i = 0$ is trivial. The case $i = 1$: since $0 < r_n < r_{n-1}$, it follows that $q_{n-1} > 1$ hence $r_{n-1} \geq 2r_n \geq 2 = f_2$. Now assume $i > 1$ and use the induction hypothesis for $i-1$ and $i-2$. Then we find

$$r_{n-i} = q_{n-i} r_{n-(i-1)} + r_{n-(i-2)} \geq r_{n-(i-1)} + r_{n-(i-2)} \geq f_i + f_{i-1} = f_{i+1}.$$

This completes the induction.

In particular, this shows that $b = r_1 \geq f_n$.

To complete the proof of the theorem, we again use induction to show $f_{5i+1} > 10^i$ for $i \geq 1$. For $i = 1$ this is correct, since $f_6 = 13 > 10$. Assuming the inequality for $i \geq 1$, we obtain

$$\begin{aligned}
f_{5(i+1)+1} &= f_{5i+6} = f_{5i+5} + f_{5i+4} = f_{5i+4} + 2f_{5i+3} + f_{5i+2} = f_{5i+3} + 3f_{5i+2} + 3f_{5i+1} + f_{5i} \\
&= f_{5i+2} + 7f_{5i+1} + 4f_{5i} = 8f_{5i+1} + 5f_{5i} > 8f_{5i+1} + 2f_{5i} + 2f_{5i-1} = 10f_{5i+1} \\
&> 10\cdot 10^i = 10^{i+1}.
\end{aligned}$$

Now write $n = 5m + k$ with $1 \leq k \leq 5$. Then $b \geq f_n \geq f_{5m+1} > 10^m$. This shows that the number of decimal digits of b is at least $m+1 \geq n/5$, so n is at most 5 times the number of decimal digits of b, which is what we wanted to prove. ■

One can use the Euclidean algorithm to construct solutions in integers of certain linear equations. This application will also be quite useful in Chapter II.

I.1.12 Theorem. (Bachet-Bézout; named after the French mathematicians Claude Gaspard Bachet 1581–1638 and Étienne Bézout 1730–1783.) *For $a,b \in \mathbb{Z}$ there exist integers $x,y \in \mathbb{Z}$ such that $ax + by = \gcd(a,b)$.*

Proof. In case $ab = 0$ this is immediate from the definition of the gcd. Now assume $a \neq 0 \neq b$. Write $r_0 = |a|$ and $r_1 = |b|$ and let n be the number of divisions with remainder computed during the execution of the Euclidean algorithm. Then we have a sequence

$$\begin{array}{lcll} r_0 & = & q_0 r_1 + r_2 & (0 < r_2 < r_1) \\ r_1 & = & q_1 r_2 + r_3 & (0 < r_3 < r_2) \\ r_2 & = & q_2 r_3 + r_4 & (0 < r_4 < r_3) \\ & & \vdots & \\ r_{n-2} & = & q_{n-2} r_{n-1} + r_n & (0 < r_n < r_{n-1}) \\ r_{n-1} & = & q_{n-1} r_n + 0 & \end{array}$$

as in the previous proof. The $n-1$st equality here presents $r_n = \gcd(a,b)$ as a linear combination of r_{n-1} and r_{n-2} with integer coefficients. Using the $n-2$nd equality one can write r_{n-1} as a combination of r_{n-2} and r_{n-3}, so this results in a presentation of $\gcd(a,b)$ as an integer linear combination of r_{n-2} and r_{n-3}. Working upward, one consecutively eliminates $r_{n-2}, r_{n-3}, \ldots, r_3, r_2$. What remains is a relation $\gcd(a,b) = xr_1 + yr_0$. By changing the sign of x and/or y if necessary, this yields an equality $ax + by = \gcd(a,b)$ as desired. ∎

The argument presented above is completely constructive. The next algorithm finds the greatest common divisor of $a, b \in \mathbb{Z}$, and integers x, y such that $ax+by = \gcd(a,b)$, all at the same time. In fact the algorithm does not consider the sequence $r_n, r_{n-1}, \ldots, r_0$ as discussed in the proof, but rather the sequence in reversed order $r_0, r_1, r_2, \ldots, r_n$. Namely, everytime a new r_i is computed, we also find $x_i, y_i \in \mathbb{Z}$ such that $x_i a + y_i b = r_i$. In the nth step, these x_n, y_n are the required x, y. Check for yourself that the algorithm indeed works.

```
# Here we find gcd(a,b), and write it as xa+yb.
if a=0
then x:=0; y:=1; gcd:=abs(b)
else if b=0
    then x:=1; y:=0; gcd:=abs(a)
    else # a and b are both nonzero in this case
        ro:=abs(a); xo:=sign(a); yo:=0;
        rn:=abs(b); x:=0; y:=sign(b);
        while rn<>0
            do
               q:=floor(ro/rn); help:=rn; rn:=ro-q*rn; ro:=help;
               help:=x; x:=xo-q*x; xo:=help;
               help:=y; y:=yo-q*y; yo:=help
            end do;
        gcd:=ro; x:=xo; y:=yo
    end if
end if; print(gcd,x,y);
```

I.1.13 Example. In Example I.1.10 we saw that $\gcd(1057, 315) = 7$. We now construct integers x, y such that $1057x + 315y = 7$ using the algorithm above. To this end, consider the following equalities:

$$\begin{array}{rcrcrl} 1 \cdot 1057 & + & 0 \cdot 315 & = & 1057 & \\ 0 \cdot 1057 & + & 1 \cdot 315 & = & 315 & \text{(subtract this 3 times from the previous:)} \\ 1 \cdot 1057 & + & -3 \cdot 315 & = & 112 & \text{(this one 2 times from the previous:)} \\ -2 \cdot 1057 & + & 7 \cdot 315 & = & 91 & \text{(this once from the previous:)} \\ 3 \cdot 1057 & + & -10 \cdot 315 & = & 21 & \text{(this 4 times from the previous:)} \\ -14 \cdot 1057 & + & 47 \cdot 315 & = & 7. & \end{array}$$

The solution found here is by no means the only one. If, for example, (x', y') is another solution, then one has

$$\left\langle \begin{pmatrix} 1057 \\ 315 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = 7 = \left\langle \begin{pmatrix} 1057 \\ 315 \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right\rangle$$

with respect to the standard inner product on $\mathbb{R}^2$, so the vector $\binom{x-x'}{y-y'}$ is perpendicular to $\binom{1057}{315}$. Using this it is not hard to find all solutions of the equation $1057x + 315y = 7$ in integers. ∎

It is easy to derive some further results from Theorem I.1.12:

I.1.14 Corollary. *Let $a, b \in \mathbb{Z}$ and put $d = \gcd(a, b)$. Every integer that is a divisor of a as well as of b is also a divisor of d. Vice versa, any divisor of d is a common divisor of a and b.*

Proof. Since d divides both a and b, the first property in Proposition I.1.4 shows that any divisor of d divides a and b as well.

For the other assertion, write $d = ax + by$ for certain $x, y \in \mathbb{Z}$. If $c|a$ and $c|b$ then also $c|ax + by = d$. This proves the corollary. ∎

I.1.15 Corollary. *Let $a, b \in \mathbb{Z}$. Then a, b are coprime if and only if there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$.*

Proof. If a, b are coprime, then by definition $\gcd(a, b) = 1$. So Theorem I.1.12 implies the existence of $x, y \in \mathbb{Z}$ with $ax + by = 1$.

Vice versa, if $ax + by = 1$ for some $x, y \in \mathbb{Z}$, put $d = \gcd(a, b)$. Then $d \geq 0$ and $d|a$ and $d|b$, hence $d|ax + by = 1$, so $d = 1$. ∎

I.1.16 Corollary. *For $a, b, c \in \mathbb{Z}$ with $\gcd(a, b) = 1$ the following holds: if $a|bc$, then $a|c$.*

Proof. Take $x, y \in \mathbb{Z}$ with $ax + by = 1$. If $a|bc$, then also $a|axc + byc = (ax + by)c = c$. ∎

I.2 Prime factorization

I.2.1 Definition. A *prime number* (prime) is an integer $p > 1$ whose only positive divisors are 1 and p.

I.2.2 Example. Small prime numbers such as $2, 3, 5, \ldots$ are well known. For much larger integers it is in general not easy to check whether they are prime. We know, for example, that $2^{524288} + 1$ is *not* prime, but $2^{859433} - 1$ is, and $2^{1048576} + 1$ is not. These numbers consist of 157827, 258716, and 315653 decimal digits, respectively. The exponents here are 2^{19}, the prime 859433, and 2^{20}. In 1962 the Swedish mathematician Hans Riesel found a divisor of the first number, namely $33629 \cdot 2^{21} + 1$. In January 1994 the second number was proven to be prime by the Americans David Slowinski and Paul Gage. At the moment of writing (October 2016) the largest known prime is $2^{77232917} - 1$, consisting of 23,249,425 decimal digits. This immense example was discovered in 2017 as part of the *Great Internet Mersenne Prime Search*. Regarding the number $2^{2^{20}} + 1$ given above, the American mathematicians Jeff Young and Duncan Buell showed that it is not a prime in 1987. However, at the time of writing (October 2018) nobody has been able to find a nontrivial factor. The smallest integer of the form $1 + 2^m$ for which it is unknown at present whether it is prime, is the one having $m = 2^{33}$. ∎

I.2.3 Theorem. *If p is prime, and $a, b \in \mathbb{Z}$ such that $p|ab$, then $p|a$ or $p|b$.*

Proof. Put $d = \gcd(a,p)$. Then d is positive since $p \neq 0$. Moreover d divides p. The definition of a prime therefore implies $d = 1$ or $d = p$. In the first case Corollary I.1.16 shows $p|b$. In the second case one has $p = d|a$. ∎

I.2.4 Corollary. *If p is prime, and $a_1, \ldots, a_n$ are integers such that $p|a_1a_2 \cdot \ldots \cdot a_n$, then there exists an index $1 \leq k \leq n$ such that $p|a_k$.*

Proof. This can be shown by induction with respect to n. For $n = 1$ the statement is obvious, and the case $n = 2$ is shown in Theorem I.2.3. Now take $n \geq 3$ and assume the statement for products of $< n$ factors. If $p|a_1a_2 \cdot \ldots \cdot a_n = (a_1) \cdot (a_2 \cdot \ldots \cdot a_n)$, Theorem I.2.3 implies that $p|a_1$ or $p|a_2 \cdot \ldots \cdot a_n$. In the first case we are done, and in the second case we use the induction hypothesis. ∎

Aided by the above properties of primes, we now show a result called the 'main theorem of arithmetic':

I.2.5 Theorem. (unique prime factorisation) *Every integer greater than 1 can be written as a product of primes. This product is unique up to the order of the factors.*

Proof. We first show that any $n \in \mathbb{Z}$ with $n > 1$ can be written as a product of primes. We use induction w.r.t. n: the case $n = 2$ is clear. Let $n > 2$ and suppose every integer greater than 1 and smaller than n can be written as a product of primes. If n is a prime number, we are done. If n is not prime, then we have $n = n_1n_2$ for some $1 < n_1, n_2 < n$. The induction hypothesis implies that both n_1 and n_2 are products of primes, so n is as well.

Next we show uniqueness. Suppose uniqueness does not hold, and take n to be the smallest integer > 1 allowing more than one factorisation into primes, say

$$n = p_1p_2 \cdot \ldots \cdot p_t = q_1q_2 \cdot \ldots \cdot q_s,$$

with primes p_i, q_j. Then $p_1|n = q_1q_2 \cdot \ldots \cdot q_s$. Since n allows more than one factorisation, n is not prime, hence $s, t > 1$ so in particular $n/p_1 > 1$. Corollary I.2.4 implies $p_1|q_k$ for some k. Since q_k is prime, we conclude that $p_1 = q_k$. Dividing the given factorisations by their common factor $p_1 = q_k$, it follows that n/p_1 allows two factorisations as well. This contradicts the minimality of n. Hence the theorem is proven. ∎

I.2.6 Remark. Computing the prime factorisation of a large integer N is an extremely difficult problem. This plays an important role in cryptography. Namely, one could break the widely used public key cryptosystem *RSA*, invented in 1978 by Ron Rivest, Adi Shamir and Leonard Adleman, by finding an efficient algorithm to solve the following problem: Given an integer N which factors as the product of two large prime numbers p and q, find p and q. RSA and related systems are discussed in the third year course 'Security and Coding'.

Although finding large primes is not easy, as we saw in Example I.2.2, the following result is very old.

I.2.7 Theorem. (Euclid) *There exist infinitely many primes.*

Proof. Suppose we have $n \geq 1$ pairwise different primes $p_1, \ldots, p_n$. Consider the integer $N = (p_1 \cdot p_2 \cdot \ldots \cdot p_n) + 1$. Take a prime q in the prime factorisation of N. Then q differs from each p_i, since otherwise q divides both N and $N - 1 = p_1 \cdot \ldots \cdot p_n$ hence also their difference $N - (N - 1) = 1$ which is impossible. We therefore conclude from the existence of n primes that also $n + 1$ primes exist. The result follows. ∎

There are many different proofs of Theorem I.2.7.

I.2.8 Remark. It is *not* true that N as constructed in the proof of Theorem I.2.7 is necessarily itself a prime. For example, given a finite set of *odd* primes, the product plus 1 is even (and > 2). Also, starting from the set of primes $\{2\}$, repeatedly taking 1 plus the product of the set of primes already considered yields $2, 3, 7, 43, 1807 = 13 \cdot 139$.

I.2.9 Definition. If p is prime and $a \in \mathbb{Z}$ is not equal to 0 or ± 1, then we write $v_p(a)$ for the number of times p appears in the prime factorisation of $|a|$. Moreover, we put $v_p(1) = v_p(-1) = 0$ and $v_p(0) = \infty$.

The number $v_p(a)$ is usually called the *valuation* of a at p. Since a prime factorisation is unique up to the order of the primes, $v_p(a)$ is well defined. If $a \in \mathbb{Z}$ is not zero, then the definition implies $|a| = \prod p^{v_p(a)}$. Here the product is taken over *all* primes p. Although there are infinitely many primes by Theorem I.2.7, the product is still well defined. Namely, only finitely many primes occur in the prime factorisation of $|a|$, and for all other primes p one has $v_p(a) = 0$ hence $p^{v_p(a)} = 1$.

I.2.10 Corollary. *Let a, b be integers.*

1. *For every prime p we have $v_p(ab) = v_p(a) + v_p(b)$.*
2. *We have $a|b$ if and only if every prime p satisfies $v_p(a) \leq v_p(b)$.*
3. *If a and b are not both zero, then*
$$\gcd(a, b) = \prod_{p \text{ prime}} p^{\min\{v_p(a), v_p(b)\}}.$$
4. *If $a \neq 0$ and also $b \neq 0$, then*
$$\operatorname{lcm}(a, b) = \prod_{p \text{ prime}} p^{\max\{v_p(a), v_p(b)\}}.$$
5. *If $a|c$ and $b|c$, then $\operatorname{lcm}(a, b)|c$.*
6. $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = |ab|$.

Proof. 1: The statement is true if a and/or b equals zero, since infinity plus infinity and also infinity plus any finite number equal infinity. The remaining case follows from the equality
$$|ab| = |a| \cdot |b| = \prod_p p^{v_p(a)} \cdot \prod_p p^{v_p(b)} = \prod_p p^{v_p(a) + v_p(b)}.$$
2: If $a|b$, then $b = qa$ for some $q \in \mathbb{Z}$, and the first part of the corollary implies $v_p(a) \leq v_p(a) + v_p(q) = v_p(qa) = v_p(b)$, for all primes p.

Vice versa, assume that $v_p(a) \leq v_p(b)$ for all primes p. Certainly $a|b$ if $b = 0$. In case $a = 0$ the assumption implies $v_p(b) = \infty$, hence $b = 0$, so again $a|b$. If a and b are both nonzero, then $q = \prod_p p^{v_p(b) - v_p(a)}$ is a well defined integer. By definition $|b|$ and $|qa|$ have the same prime factorisation, hence $b = \pm qa$, which implies $a|b$.
3: By assertion 2., the integers d which divide both a and b are precisely those integers with the properties $v_p(d) \leq v_p(a)$ and $v_p(d) \leq v_p(b)$ for all primes p. Since $a \neq 0$ or $b \neq 0$, we get that $\min\{v_p(a), v_p(b)\}$ is finite for all primes p, and nonzero for only finitely many primes p. So $\prod_p p^{\min\{v_p(a), v_p(b)\}}$ is a well defined integer, dividing both a and b and moreover at least as large as any other common divisor. Hence it equals $\gcd(a, b)$.
4: Suppose that a and b are both nonzero, then $v_p(a)$ and $v_p(b)$ are both finite for all primes p and nonzero for only finitely many primes p. Thus $k = \prod_p p^{\max\{v_p(a), v_p(b)\}}$ is well defined and positive. By assertion 2., k is a multiple of both a and b. Every

common multiple c satisfies$v_p(c) \geq v_p(a)$ and $v_p(c) \geq v_p(b)$ for all primes p by 2., hence k is the smallest positive common multiple, i.e. $k = \operatorname{lcm}(a,b)$.
5: If $ab = 0$ and $a|c$ and $b|c$, then $c = 0$, so in this case the assertion holds. If $ab \neq 0$, then the assertion follows by combining properties 2. and 4..
6: This holds whenever $ab = 0$, since then $\operatorname{lcm}(a,b) = 0$ by definition. For $ab \neq 0$, one has

$$\begin{aligned} |ab| &= \prod_p p^{v_p(a)} \cdot \prod_p p^{v_p(b)} = \prod_p p^{v_p(a)+v_p(b)} \\ &= \prod_p p^{\min\{v_p(a),v_p(b)\}+\max\{v_p(a),v_p(b)\}} \end{aligned}$$

which equals $\gcd(a,b) \cdot \operatorname{lcm}(a,b)$ by 3. and 4.. ■

I.2.11 Remark. The formulas given in Corollary I.2.10 provide a method to compute $\gcd(a,b)$ and $\operatorname{lcm}(a,b)$ if the prime factorisation of a and b is known. In general this might not be the case, and factoring an integer is typically much more difficult than directly computing $\gcd(a,b)$ with the Euclidean algorithm. The knowledge of $\gcd(a,b)$ allows us to find $\operatorname{lcm}(a,b)$ by the formula $\operatorname{lcm}(a,b) = |ab|/\gcd(a,b)$. Note that we used unique prime factorisation to *prove* this formula. However, this factorisation is no longer needed to compute $\operatorname{lcm}(a,b)$ by means of the formula.

I.3 Exercises

1. ('The b-ary system'). Let $a, b \in \mathbb{Z}$ with $a \geq 1$ and $b \geq 2$. Show that there exist a positive integer t and integers $c_0, c_1, \ldots, c_t \in \{0, 1, \ldots, b-1\}$ such that $c_t \neq 0$ and

$$a = c_t b^t + \ldots + c_2 b^2 + c_1 b + c_0.$$

   Show that these $t, c_0, \ldots, c_t$ are *unique*.
2. Prove that if $a, b \in \mathbb{Z}$ are not both equal to zero, then $a/\gcd(a,b)$ and $b/\gcd(a,b)$ are coprime.
3. Determine $d = \gcd(3354, 3081)$ and find $x, y \in \mathbb{Z}$ with $3354x + 3081y = d$. Next, find *all* solutions of this equation in integers.
4. Let $(f_n)_{n \geq 0}$ be the Fibonacci sequence defined by $f_0 = f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$. How many divisions with remainder are performed by the Euclidean algorithm when computing $\gcd(f_n, f_{n+1})$? Show that $\gcd(f_n, f_{n+1}) = 1$ for all $n \geq 0$.
5. Let $n \in \mathbb{Z}$, $n \geq 2$. Show that n is prime if and only if n has no divisor d such that $1 < d \leq \sqrt{n}$.
6. Prove that infinitely many primes exist which leave a remainder 3 upon division by 4.
7. Prove that infinitely many primes p exist with the property that $p - 2$ is not prime.
8. Prove that for $a, b, c \in \mathbb{Z}$ one has:
   (a) If $\gcd(a,b) = \gcd(a,c) = 1$, then $\gcd(a, bc) = 1$.
   (b) If a and b are coprime and both divide c, then their product divides c.
   (c) If $c \geq 0$, then $\gcd(ac, bc) = c \cdot \gcd(a,b)$.
9. Let $a, b \in \mathbb{Z}$ both be positive.
   (a) Let r be the remainder upon dividing a by b. Show that $2^r - 1$ is the remainder upon dividing $2^a - 1$ by $2^b - 1$.
   (b) Show that $2^b - 1 | 2^a - 1$ if and only if $b | a$.
   (c) Prove that $\gcd(2^a - 1, 2^b - 1) = 2^{\gcd(a,b)} - 1$.
   (d) Are the above assertions still true if we replace '2' by some integer $c > 2$?
10. Let $a, b, n \in \mathbb{Z}$ such that $n \geq 0$.
    (a) Show that $a - b | a^n - b^n$.
    (b) Show that if n is odd, then $a + b | a^n + b^n$.
    (c) Now take $b = 1$ and suppose $a > 1$ and $n > 1$. Prove that if $a^n - 1$ is prime, then $a = 2$ and n is prime.
    (d) Take $a = 2$ and $n = 11$ and verify that $a^n - 1$ is not prime. So apparently the converse of the property above does not necessarily hold.
11. Show (using part (b) of the previous exercise) that if $2^n + 1$ is prime, then n is of the form $n = 2^k$ for some integer $k \geq 0$.
12. Prove that if $n^4 + 4^n$ is prime for some positive integer n, then $n = 1$.
13. (a) Show that any prime $p > 3$ satisfies $24 | p^2 - 1$.
    (b) Show that if p_1, p_2, p_3, p_4, p_5 are (not necessarily distinct) primes, and $p_1 p_2 p_3 p_4 p_5 + 1 = p^2$ for some prime p, then $p = 7$ or $p = 11$ or $p = 13$.
14. Suppose that $p_1, \ldots, p_n$ are pairwise distinct primes. Show that the real numbers $\log(p_1), \ldots, \log(p_n)$ are linearly independent over the rationals. In other words, show that if $a_1, \ldots, a_n \in \mathbb{Q}$ satisfy $a_1 \log(p_1) + \ldots + a_n \log(p_n) = 0$, then $a_1 = \ldots = a_n = 0$.

II MODULAR ARITHMETIC

In this chapter we develop the basic properties of calculations with remainders upon division.

II.1 Residue classes modulo N

Let N be an arbitrary positive integer.

II.1.1 Definition. Two integers a, b are called *congruent modulo* N if $N \mid a-b$. This is denoted by $a \equiv b \bmod N$. We call N the *modulus*.

Integers a, b are congruent modulo N precisely when they have the same remainder upon division by N. Indeed, if $a = q_1N + r_1$ and $b = q_2N + r_2$ for certain $0 \le r_1, r_2 < N$, then the statement $N \mid a-b$ is equivalent to $N \mid r_1 - r_2$. Since $r_1 - r_2$ is strictly between $-N$ and $+N$, we have $N \mid r_1 - r_2$ if and only if $r_1 - r_2 = 0$, i.e., $r_1 = r_2$.

We now discuss some properties of the relation 'being congruent modulo N.

II.1.2 Lemma. *Let a, b, c be integers. Then the following assertions hold.*

1. *(Reflexivity) We have $a \equiv a \bmod N$.*
2. *(Symmetry) We have $a \equiv b \bmod N$ if and only if $b \equiv a \bmod N$.*
3. *(Transitivity) If $a \equiv b \bmod N$ and $b \equiv c \bmod N$, then $a \equiv c \bmod N$.*

Proof. The proof follows directly from the definition of congruence. ∎

A relation satisfying the assertions of Lemma II.1.2 is called an *equivalence relation*. We will see further examples of this later on in the course.

An equivalence relation partitions a set into a union of pairwise disjoint subsets. In our case these subsets are called residue classes modulo N. Explicitly:

II.1.3 Definition. For $a \in \mathbb{Z}$ the *residue class of a modulo N* is defined as

$$\{b \in \mathbb{Z} \mid b \equiv a \bmod N\}.$$

We denote this residue class by $a \bmod N$. If the modulus N is clear from the context, we also write $\overline{a}$ for $a \bmod N$. If $b \in A \bmod N$, then we call b a *representative* for $a \bmod N$.

By definition $a \bmod N$ is a *subset* of $\mathbb{Z}$. If $a = qN + r$, then $a \equiv r \bmod N$, and the residue class $r \bmod N$ equals $a \bmod N$. Hence there are as many distinct residue classes modulo N as there are possible remainders upon division by N, namely N such classes. The residue class of a modulo N consists of all integers of the form $a + Nk$ for some $k \in \mathbb{Z}$, hence we can also write

$$a \bmod N = a + N\mathbb{Z}.$$

II.1.4 Example. As explained above, there are 4 distinct residue classes for $N = 4$, namely 0 mod 4 and 1 mod 4 and 2 mod 4 and 3 mod 4. These are pairwise disjoint subsets of $\mathbb{Z}$ whose union is all of $\mathbb{Z}$. They are:

$$\begin{aligned}
\{ \quad &\ldots\ldots, -284, \ldots, -8, -4, 0, 4, \ldots, 1016, \ldots\ldots \quad \} = 0 \bmod 4, \\
\{ \quad &\ldots\ldots, -283, \ldots, -7, -3, 1, \ldots, 1017, \ldots\ldots \quad \} = 1 \bmod 4, \\
\{ \quad &\ldots\ldots, -282, \ldots, -6, -2, 2, \ldots, 1018, \ldots\ldots \quad \} = 2 \bmod 4, \\
\{ \quad &\ldots\ldots, -281, \ldots, -5, -1, 3, \ldots, 1019, \ldots\ldots \quad \} = 3 \bmod 4.
\end{aligned}$$

The residue class 17 mod 4 equals 1 mod 4. Using the notation $\overline{a} = a \bmod 4$ this is expressed as $\overline{17} = \overline{1}$. Similarly one has $\overline{-1001} = \overline{3}$. ▬■

The following elementary property is an immediate consequence of generalities concerning equivalence relations. Nevertheless we present a proof, illustrating how the given definitions are used.

II.1.5 Lemma. *For $a, b \in \mathbb{Z}$ one has $a \bmod N = b \bmod N$ if and only if $a \equiv b \bmod N$.*

Proof. Assume $a \bmod N = b \bmod N$. Since a is an element of the residue class $a \bmod N = b \bmod N$, we must have $a \equiv b \bmod N$.

Vice versa, suppose $a \equiv b \bmod N$. As we saw, this means that a and b yield the same remainder upon division by N. Hence an integer c is in the residue class $a \bmod N$ if and only if c and a yield the same remainder upon division by N, which is equivalent to c leaving the same remainder as b, therefore to c being in $b \bmod N$. This shows $a \bmod N = b \bmod N$. ■

II.1.6 Theorem. *Let $\overline{a_1}, \overline{a_2}, \overline{b_1}, \overline{b_2}$ be residue classes modulo N, where a_1, a_2, b_1, b_2 are integers. Suppose $\overline{a_1} = \overline{a_2}$ and $\overline{b_1} = \overline{b_2}$. Then*

$$\overline{a_1 + b_1} = \overline{a_2 + b_2} \quad \text{and} \quad \overline{a_1 b_1} = \overline{a_2 b_2}.$$

Proof. From $\overline{a_1} = \overline{a_2}$ and $\overline{b_1} = \overline{b_2}$ it follows by Lemma II.1.5 that $q, q' \in \mathbb{Z}$ exist with $a_2 = a_1 + Nq$ and $b_2 = b_1 + Nq'$. Hence $a_2 + b_2 = a_1 + b_1 + N(q + q')$, which by Lemma II.1.5 implies $\overline{a_1 + b_1} = \overline{a_2 + b_2}$. Furthermore,

$$a_2 b_2 = (a_1 + Nq)(a_2 + Nq') = a_1 a_2 + N(a_1 q' + q a_2 + Nqq'),$$

and the same reasoning implies $\overline{a_1 b_1} = \overline{a_2 b_2}$. ■

II.1.7 Definition. (*Adding and multiplying residue classes.*) We denote the set of residue classes modulo N by $\mathbb{Z}/N\mathbb{Z}$ (or $\mathbb{Z}/(N)$). For $a \bmod N, b \bmod N \in \mathbb{Z}/N\mathbb{Z}$ we define

$$(a \bmod N) + (b \bmod N) := (r_1 + r_2) \bmod N$$

and

$$(a \bmod N) \cdot (b \bmod N) := r_1 r_2 \bmod N,$$

with r_1 an arbitrary element of $a \bmod N$ and r_2 an arbitrary element of $b \bmod N$. Theorem II.1.6 shows that the resulting residue classes are independent of the choice of r_1, r_2, which implies that $(a \bmod N) + (b \bmod N)$ and $(a \bmod N) \cdot (b \bmod N)$ are well-defined.

II.1.8 Remark. If one chooses $r_1 = a$ and $r_2 = b$ in the above definition (which is allowed, since a, b are elements of $a \bmod N, b \bmod N$, respectively), then the definition reads $\overline{a} + \overline{b} := \overline{a+b}$ and $\overline{a} \cdot \overline{b} := \overline{ab}$.

II.1.9 Example. Take $N = 17$. Then $\overline{-1} = \overline{67}$, since the difference of -1 and 67 is divisible by 17. Hence we get $\overline{1} = \overline{(-1)(-1)} = \overline{-1} \cdot \overline{-1} = \overline{67} \cdot \overline{67} = \overline{67^2}$. Apparently, 67^2 and 1 yield the same remainder upon division by 17, or in other words, $67^2 - 1$ is divisible by 17. (Of course this could be seen without residue classes: $67^2 - 1 = (67+1)(67-1)$.) ∎

Since we can add and multiply residue classes modulo N, we can also raise them to a positive power n.

II.1.10 Definition. For a natural number n the nth power of a residue class $\overline{a}$, denoted by $\overline{a}^n$, is inductively defined as follows. We define $\overline{a}^1 := \overline{a}$, and if we have defined $\overline{a}^n$ for $n \geq 1$, then we set $\overline{a}^{n+1} := \overline{a}^n \cdot \overline{a}$.

It holds that $\overline{a}^m = \overline{a^m}$ and $\overline{a}^{n+m} = \overline{a}^n \cdot \overline{a}^m$, as is easily verified using mathematical induction with respect to m. Moreover $\overline{ab}^m = \overline{(ab)^m} = \overline{a^m b^m} = \overline{a^m} \cdot \overline{b^m} = \overline{a}^m \cdot \overline{b}^m$.

II.1.11 Example. To illustrate the use of these definitions, we will show that the integer $2^{1000} + 1$ is divisible by 257. Write $\overline{a}$ for the residue class of a modulo 257. Then

$$\overline{2^{1000}} = \overline{(2^8)^{125}} = \overline{256}^{125} = \overline{-1}^{125} = \overline{-1}.$$

Since 2^{1000} and -1 yield the same residue class modulo 257, their difference is divisible by 257 which is what we wanted to show. Note that $2^{1000} + 1$ has 302 decimal digits, so to check the asserted divisibility using a simple division by 257 is quite elaborate. ∎

II.1.12 Example. We calculate the last two decimal digits of 2^{1000}. This is the same as finding the remainder of 2^{1000} upon division by 100. In $\mathbb{Z}/100\mathbb{Z}$ we have

$$\overline{16}^6 = \overline{4}^6 \cdot \overline{4}^6 = \overline{4096} \cdot \overline{4096} = \overline{-4} \cdot \overline{-4} = \overline{16},$$

since $4^6 = 2^{12} = 4096$. Furthermore $1000 = 4 \cdot 250$ and $250 = 6 \cdot 41 + 4$ and $41 = 6 \cdot 6 + 5$, so

$$\begin{aligned}
\overline{2^{1000}} = \overline{2^4}^{250} &= \overline{16}^4 \cdot (\overline{16}^6)^{41} \\
&= \overline{16}^4 \cdot \overline{16}^{41} \qquad = \overline{16}^4 \cdot (\overline{16}^6)^6 \cdot \overline{16}^5 \\
&\qquad\qquad\quad = \overline{16}^4 \cdot \overline{16} \cdot \overline{16}^5 \\
&\qquad\qquad\quad = \overline{16}^4 \cdot \overline{16}^6 = \overline{16}^4 \cdot \overline{16} \\
&\qquad\qquad\quad = \overline{16}^5 = (\overline{2^{10}})^2 = \overline{24}^2 = \overline{76}.
\end{aligned}$$

Hence the desired two decimals are 76. ∎

II.2 Units modulo N

II.2.1 Definition. A residue class $a \bmod N$ is called a *unit modulo* N (or *invertible modulo* N) if there exists a residue class $b \bmod N$ such that $(a \bmod N) \cdot (b \bmod N) = 1 \bmod N$.

The subset of $\mathbb{Z}/N\mathbb{Z}$ consisting of all units modulo N is denoted as $(\mathbb{Z}/N\mathbb{Z})^\times$.

II.2.2 Example. Take $N = 12$. Then $\mathbb{Z}/12\mathbb{Z} = \{\overline{0}, \overline{1}, \ldots, \overline{10}, \overline{11}\}$. For each of these classes we check whether it is in $(\mathbb{Z}/12\mathbb{Z})^\times$. If $a, b \in \mathbb{Z}$ and $\overline{a} \cdot \overline{b} = \overline{1}$, this means that $ab = 1 + 12k$ for certain $k \in \mathbb{Z}$. In particular, if $\overline{a}$ is a unit modulo 12, then a is not divisible by 2, nor by 3. Hence

$$(\mathbb{Z}/12\mathbb{Z})^\times \subset \{\overline{1}, \overline{5}, \overline{7}, \overline{11}\}.$$

Since $\overline{1}^2 = \overline{5}^2 = \overline{7}^2 = \overline{11}^2 = \overline{1}$, the four given classes are indeed units modulo 12. So

$$(\mathbb{Z}/12\mathbb{Z})^\times = \{\overline{1}, \overline{5}, \overline{7}, \overline{11}\}.$$

—■

We present a simple criterion for finding the units modulo N.

II.2.3 Theorem. *Let $a \in \mathbb{Z}$. Then $a \bmod N \in (\mathbb{Z}/N\mathbb{Z})^\times$ if and only if $\gcd(a, N) = 1$.*

Proof. Let $a \in \mathbb{Z}$. By definition the assertion '$a \bmod N \in (\mathbb{Z}/N\mathbb{Z})^\times$' means that there exists a $b \in \mathbb{Z}$ such that

$$(ab) \bmod N = (a \bmod N) \cdot (b \bmod N) = 1 \bmod N.$$

This is equivalent to the existence of $q, b \in \mathbb{Z}$ such that $ab - 1 = Nq$, which is the same as $ab - Nq = 1$. In Corollary I.1.15 it was shown that such integers exist if and only if $\gcd(a, N) = 1$. ■

II.2.4 Definition. (Euler's totient function (or Euler's phi function); Leonhard Euler, Swiss mathematician, 1707–1783) The number of elements of $(\mathbb{Z}/N\mathbb{Z})^\times$ is denoted by $\varphi(N)$.

II.2.5 Corollary. *The number $\varphi(N)$ equals the number of integers $a \in \mathbb{Z}$ with $1 \le a \le N$ and $\gcd(a, N) = 1$. In particular, a positive integer p is prime if and only if $\varphi(p) = p - 1$.*

Proof. This is immediate from the definitions and from Theorem II.2.3. ■

We list some properties of $(\mathbb{Z}/N\mathbb{Z})^\times$.

II.2.6 Theorem. *1. If $a \bmod N$ and $b \bmod N$ are units modulo N, then so is their product.*
2. If $a \bmod N \in (\mathbb{Z}/N\mathbb{Z})^\times$, then a residue class $b \bmod N$ such that $(a \bmod N) \cdot (b \bmod N) = 1 \bmod N$ is also a unit modulo N.
3. For each $a \bmod N \in (\mathbb{Z}/N\mathbb{Z})^\times$ there is a unique class $b \bmod N \in (\mathbb{Z}/N\mathbb{Z})^\times$ with $(a \bmod N) \cdot (b \bmod N) = 1 \bmod N$.

Proof. 1: By Theorem II.2.3 the assertion is equivalent to the following: if both $\gcd(a, N) = 1$ and $\gcd(b, N) = 1$, then also $\gcd(ab, N) = 1$. To see why the latter implication holds, assume $\gcd(ab, N) \neq 1$. Then a prime p dividing $\gcd(ab, N)$ exists. This prime divides N and ab, hence by Theorem I.2.3 we have $p \mid a$ or $p \mid b$. This contradicts $\gcd(a, N) = \gcd(b, N) = 1$.

Alternative proof: The classes $\overline{a}, \overline{b}$ are units modulo N, so there exist $\overline{c}, \overline{d}$ such that $\overline{c} \cdot \overline{a} = \overline{db} = \overline{1}$. Put $\overline{e} = \overline{d} \cdot \overline{c}$, then

$$\overline{e} \cdot \overline{ab} = \overline{dcab} = \overline{d} \cdot \overline{ca} \cdot \overline{b} = \overline{db} = \overline{1},$$

hence $\overline{ab} \in (\mathbb{Z}/N\mathbb{Z})^\times$.
2: This is immediate from the fact that $\overline{a} \cdot \overline{b} = \overline{ab} = \overline{ba} = \overline{b} \cdot \overline{a}$.
3: If $\overline{a} \cdot \overline{b_1} = \overline{1} = \overline{a} \cdot \overline{b_2}$, then also

$$\overline{b_1} = \overline{b_1} \cdot \overline{1} = \overline{b_1} \cdot \overline{ab_2} = \overline{b_1 ab_2} = \overline{ab_1} \cdot \overline{b_2} = \overline{b_2}.$$

This proves Theorem II.2.6. ■

II.2.7 Definition. For $\overline{a} \in (\mathbb{Z}/N\mathbb{Z})^\times$ the unique class $\overline{b} \in (\mathbb{Z}/N\mathbb{Z})^\times$ with the property $\overline{a} \cdot \overline{b} = \overline{1}$ is called the *inverse* of $\overline{a}$. We denote it by $\overline{a}^{-1}$.

II.2.8 Remark. For $a \in \mathbb{Z}$ with $a \bmod N \in (\mathbb{Z}/N\mathbb{Z})^\times$, the inverse of $a \bmod N$ may be found using the Euclidean algorithm. Namely $a \bmod N \in (\mathbb{Z}/N\mathbb{Z})^\times$ being a unit implies $\gcd(a,N) = 1$. So there exist $x, y \in \mathbb{Z}$ with $xa + yN = 1$, and any such x satisfies $\overline{x} \cdot \overline{a} = \overline{1}$, in other words, $x \bmod N$ is the inverse of $a \bmod N$.

Besides addition, subtraction and multiplication, the most important operations on residue classes modulo N that we have encountered so far are the operations exponentiation (i.e. raising to a positive power) and, in the special case of units modulo N, taking the inverse.

In computer algebra systems such as MAGMA, Maple, Mathematica, and PARI, and even in WolframAlpha, standard routines are implemented for the mentioned operations. For example this may look as follows in Maple:

```
100^(-1) mod 420001;
7 &^ (420!) mod 100;
```

The symbol & in the second line makes sure that Maple does not first raise 7 to the power 420!, and subsequently find the remainder of the result upon division by 100. Instead, a far more efficient way to obtain the answer is used.

II.2.9 Example. We have $(13 \bmod 37)^{-1} = 20 \bmod 37$ (check this!). ∎

II.2.10 Theorem. (Euler) *For all $a \bmod N \in (\mathbb{Z}/N\mathbb{Z})^\times$ one has*

$$(a \bmod N)^{\varphi(N)} = 1 \bmod N.$$

An alternative way to formulate the theorem is the following: if $a, N \in \mathbb{Z}$ satisfy $N > 0$ and $\gcd(a,N) = 1$, then $N \mid a^{\varphi(N)} - 1$. The theorem and some of its consequences are crucial for setting up the crypto system RSA, see Remark I.2.6. It is also useful for testing whether a (very large) integer is prime. We return to this below.

Proof. Write $(\mathbb{Z}/N\mathbb{Z})^\times = \{\overline{a_1}, \ldots, \overline{a_{\varphi(N)}}\}$. In Theorem II.2.6 we saw that a product of units modulo N is a unit as well, hence

$$\epsilon := \overline{a_1} \cdot \overline{a_2} \cdot \ldots \cdot \overline{a_{\varphi(N)}}$$

is a unit modulo N.

Consider the map 'multiplication by $\overline{a}$'. This map sends $(\mathbb{Z}/N\mathbb{Z})^\times$ to itself, since $\overline{a}$ is a unit. We denote this map by ψ, i.e.

$$\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times ;\ \overline{b} \mapsto \overline{a}\overline{b}.$$

We will now show that ψ is a bijection. First we show injectivity. Indeed, let $\overline{b}, \overline{c} \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $\overline{a} \cdot \overline{b} = \overline{a} \cdot \overline{c}$. Now multiplying both products with the inverse of $\overline{a}$ yields $\overline{b} = \overline{c}$. Hence the map is injective. This means that distinct elements are mapped to distinct elements, and therefore the image has the same number elements as the source, namely exactly $\varphi(N)$. As a consequence the map is surjective as well. This means

$$(\mathbb{Z}/N\mathbb{Z})^\times = \{\overline{a} \cdot \overline{a_1}, \overline{a} \cdot \overline{a_2}, \ldots, \overline{a} \cdot \overline{a_{\varphi(N)}}\}.$$

As we saw, multiplying all of these elements yields ϵ. On the other hand this product equals

$$(\overline{a} \cdot \overline{a_1}) \cdot (\overline{a} \cdot \overline{a_2}) \cdot \ldots \cdot (\overline{a} \cdot \overline{a_{\varphi(N)}}) = \overline{a}^{\varphi(N)} \cdot \overline{a_1} \cdot \overline{a_2} \cdot \ldots \cdot \overline{a_{\varphi(N)}} = \overline{a}^{\varphi(N)} \cdot \epsilon.$$

This shows $\epsilon = \overline{a}^{\varphi(N)} \cdot \epsilon$, and multiplying by the inverse of ϵ now gives $\overline{a}^{\varphi(N)} = \overline{1}$, which is what we wanted to prove. ∎

II.2.11 Corollary. (Fermat's little theorem; Pierre de Fermat, French amateur mathematician, 1601–1665) *If p is prime, then $(a \bmod p)^p = a \bmod p$ for every $a \in \mathbb{Z}$.*

Proof. If $a \bmod p$ is not a unit modulo p, then $a \bmod p = 0 \bmod p$ because p is prime. In this case the corollary is clear. Now note that since p is prime, $\varphi(p) = p-1$ by Corollary II.2.5. Hence for $a \bmod p \in (\mathbb{Z}/p\mathbb{Z})^\times$ one finds using Theorem II.2.10 that

$$(a \bmod p)^p = (a \bmod p)\cdot(a \bmod p)^{p-1} = (a \bmod p)\cdot\overline{1} = a \bmod p.$$

■

Fermat's little theorem yields an efficient and simple criterion which can often be used to check that certain large integers are not prime. Namely, to test whether N is prime, take (for example) $a = 2$, and calculate $(2 \bmod N)^N$. If the result is not $2 \bmod N$, then Fermat's little theorem shows that N is not prime. Here the exponentiation can be done by computing in the order of $\log(N)$ multiplications/divisions with remainder. Moreover, this involves only integers between 0 and N. Hence this 'compositeness test' is much faster than simply testing whether N has some divisor between 1 and $\sqrt{N}$. However, our algorithm yields not as much information. For instance, if one concludes from the algorithm that N is not prime, this does not provide any relevant information concerning possible divisors of N.

A more serious problem arises because there exist composite integers N such that $(2 \bmod N)^N = 2 \bmod N$. For instance, $341 = 11 \cdot 31$ has this property. Hence we have to modify the test, by computing $3 \bmod 341$ instead of $2 \bmod 341$; indeed $(3 \bmod 341)^{341} = 168 \bmod 341$. Unfortunately, there are composite integers for which the test will never prove that they are not prime; the so-called *Carmichael numbers*. These are composite positive integers N with the property that every $a \in \mathbb{Z}$ satisfies $(a \bmod N)^N = a \bmod N$. The smallest Carmichael number is $N = 561 = 3 \cdot 11 \cdot 17$. In 1992 Alford, Granville and Pomerance proved that infinitely many such Carmichael numbers exist. A version of the above test which also works for Carmichael numbers, and hence can be used to show that a number is prime, is due to Lucas.

II.3 The Chinese remainder theorem

To check that $N = 561 = 3 \cdot 11 \cdot 17$ is indeed a Carmichael number, namely that every integer a satisfies $561 \mid a^{561} - a$, one has to test divisibility by $561 = 3 \cdot 11 \cdot 17$. To simplify this, it is natural to test divisibility by 3, 11, and 17 instead. In this way the property can be checked rather simply, as follows. If a is coprime to 3, Theorem II.2.10 shows that $(a \bmod 3)^2 = 1 \bmod 3$, hence any *even* exponent $m = 2k$ satisfies $(a \bmod 3)^m = (1 \bmod 3)^k = 1 \bmod 3$, i.e., $3 \mid a^{2k} - 1$. Multiplying by a then shows $3 \mid a^{2k+1} - a$. Clearly this also holds for any a which is divisible by 3, and hence for all $a \in \mathbb{Z}$.

In the same way one shows that any exponent $m = 10\ell$ which is a multiple of 10 satisfies $11 \mid a^{10\ell} - 1$, provided a is coprime to 11. As a consequence, every $a \in \mathbb{Z}$ has the property $11 \mid a^{10\ell+1} - a$. Finally, a similar reasoning shows $17 \mid a^{16n+1} - a$ for all $a \in \mathbb{Z}$. Combining the three divisibility properties above, one concludes that any exponent m equal to 1 plus a multiple of each of 16, 10, and 2, in other words any exponent of the form

$$m = 1 + k\,\mathrm{lcm}(16, 10, 2) = 1 + 80k\,,\ k \in \mathbb{Z},$$

satisfies $3 \cdot 11 \cdot 17 = 561 \mid a^m - a$ for all $a \in \mathbb{Z}$. In particular the exponent $561 = 1 + 80 \cdot 7$ has the required form, so we conclude $561 \mid a^{561} - a$.

The above result was obtained by combining two ingredients: arithmetic modulo a prime p (implying the required special case of Theorem II.2.10) on the one hand, and deducing divisibility by a product of primes from from divisibility by the primes. We now consider the second ingredient in greater generality; in particular, not only for primes. For this we need to check when it is possible to deduce information modulo one integer from information modulo another. We phrase it in terms of a map.

II.3.1 Lemma. *Suppose $N, M \in \mathbb{Z}$ are positive. The rule 'send the residue class of a modulo N to the residue class of a modulo M' yields a well defined map: $\mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{Z}/M\mathbb{Z}$ if and only if $M \mid N$.*

Proof. We have to examine under which condition(s) the rule described in the lemma defines a map. Consider a residue class $a \bmod \mathbb{Z}$ in $\mathbb{Z}/N\mathbb{Z}$, of $a \in \mathbb{Z}$. It is also the residue class of $a+N, a+2N, a-N$, and, more generally, of any representative of $a \bmod \mathbb{Z}$, i.e. of any $b \in \mathbb{Z}$ with $N \mid a-b$. The desired map would send this class to $a \bmod M$, but also, for any representative b to $b \bmod M$. So in order to have a well-defined map, we need that the assignment is *independent* of the choice of representative of the residue class. If this is satisfied, then we also get a well-defined map. Concretely, the assignment is well-defined if and only if we have $a \bmod M = b \bmod M$ for all $a, b \in \mathbb{Z}$ such that $N \mid a-b$. In other words: $N \mid a-b$ should imply $M \mid a-b$. We need this condition to be satisfied for all $a, b \in \mathbb{Z}$. Taking $a = N$ and $b = 0$ shows that the condition $M \mid N$ is necessary. Vice versa, if $M \mid N$, then one obtains $M \mid N \mid a-b$ for all $a, b \in \mathbb{Z}$ with $N \mid a-b$, hence in particular $M \mid a-b$. This proves the lemma. ∎

II.3.2 Remark. Make sure that you understand the statement and proof of Lemma II.3.1, because quite a few results and proofs in this course will follow a similar pattern. It may appear somewhat strange at first sight, but it reveals a very essential property of modular arithmetic. Namely, residue classes are sets, and if one picks an element from such a set and performs certain operations on it, the final result may very well change if a different element from the same residue class was chosen. In this proof (as well as in several others to come) the main task is to show that the operations under consideration are independent of the chosen representative. We have already seen another example of this principle – to show that Definition II.1.7 makes sense, we first needed Theorem II.1.6, which proved that the operations addition and multiplication on $\mathbb{Z}$ descend to well-defined operations on $\mathbb{Z}/N\mathbb{Z}$.

II.3.3 Example. We consider $N = 4$. Then $a \bmod 4 \mapsto a \bmod 2$ defines a map from $\mathbb{Z}/4\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$. The image of each of $1 \bmod 4$ and $3 \bmod 4$ is $1 \bmod 2$, and both $0 \bmod 4$ and $2 \bmod 4$ have image $0 \bmod 2$. On the other hand $a \bmod 2 \mapsto a \bmod 4$ is *not* well defined. For example, $1 \bmod 2$ and $3 \bmod 2$ are the same residue class, but $1 \bmod 4$ and $3 \bmod 4$ differ.

Put differently, this example shows: if we know the remainder upon division by 4 of some integer, then we also know its remainder upon division by 2. On the other hand, given the remainder upon division by 2 one cannot deduce remainder upon division by 4. ∎

In order to formulate the main result of this section, we briefly recall a notation from basic set theory. Given sets V and W, one denotes the Cartesian product of V and W bt $V \times W$. By definition this consists of all ordered pairs consisting of an element from V followed by an element from W:

$$V \times W := \{(v, w) \mid v \in V \text{ and } w \in W\}.$$

If V and W are finite, we get $|V \times W| = |V| \cdot |W|$.

II.3.4 Theorem. (The Chinese remainder theorem)
Let N, M be positive integers with $\gcd(N,M) = 1$. *The assignment*

$$a \bmod NM \mapsto (a \bmod N, a \bmod M) : \mathbb{Z}/NM\mathbb{Z} \longrightarrow \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$$

defines a well-defined map. This map is bijective.

Moreover it maps $(\mathbb{Z}/NM\mathbb{Z})^\times$ to $(\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/M\mathbb{Z})^\times$, and this is a bijection as well.

Proof. Since both N and M divide NM, the assignments $a \bmod NM \mapsto a \bmod N$ is well-defined by Lemma II.3.1, and the same holds for N replaced by M. Hence we get a well-defined map $\psi : \mathbb{Z}/NM\mathbb{Z} \longrightarrow \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$.

Next we show that ψ is injective. If $a, b \in \mathbb{Z}$ such $(a \bmod N, a \bmod M) = (b \bmod N, b \bmod M)$, this means $N \mid a-b$ and $M \mid a-b$ by definition. Now Corollary I.2.10 implies that $\operatorname{lcm}(N,M) \mid a-b$. Furthermore, since $\gcd(N,M) = 1$, one obtains $NM = \gcd(N,M) \cdot \operatorname{lcm}(N,M) = \operatorname{lcm}(N,M)$ from the same corollary. We conclude that $NM \mid a-b$, i.e., $a \bmod NM = b \bmod NM$. The map ψ is therefore injective, and since both $\mathbb{Z}/NM\mathbb{Z}$ and $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$ consist of NM elements, ψ must be surjective as well. Hence it is bijective.

If $a \in \mathbb{Z}$ satisfies $a \bmod NM \in (\mathbb{Z}/NM\mathbb{Z})^\times$, then $\gcd(a, NM) = 1$, hence in particular a $\gcd(a,N) = 1 = \gcd(a,M)$. This means precisely that $(a \bmod N, a \bmod M) \in (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/M\mathbb{Z})^\times$. So ψ sends $(\mathbb{Z}/NM\mathbb{Z})^\times$ to $(\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/M\mathbb{Z})^\times$.

Vice versa, let $(a \bmod N, b \bmod M) \in (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/M\mathbb{Z})^\times$, where $a, b \in \mathbb{Z}$. It follows that $\gcd(a,N) = 1 = \gcd(b,N)$. Since ψ is surjective, there exists $c \in \mathbb{Z}$ with $(c \bmod N, c \bmod M) = (a \bmod N, b \bmod M)$. This means $c = q_1 N + a$ and $c = q_2 M + b$, so according to Lemma I.1.9 one concludes $\gcd(c,N) = \gcd(a,N) = 1$ and $\gcd(c,M) = \gcd(b,M) = 1$. As a consequence $\gcd(c,NM) = 1$, i.e., $c \bmod NM \in (\mathbb{Z}/NM\mathbb{Z})^\times$. So the image of the restriction of ψ to $(\mathbb{Z}/NM\mathbb{Z})^\times$ is $(\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/M\mathbb{Z})^\times$, and clearly this restriction is injective as well. This proves the Chinese remainder theorem. ∎

II.3.5 Remark. The Chinese remainder theorem can be viewed as a solvability criterion for systems of simultaneous congruences: given a, b, N, M one asks for $x \in \mathbb{Z}$ satisfying both $x \equiv a \bmod N$ and $x \equiv b \bmod M$. The theorem states that a solution x always exists whenever $\gcd(N,M) = 1$, and, moreover, that the set of all solutions is a residue class modulo NM.

For instance, let us find all $x \in \mathbb{Z}$ with $x \equiv 4 \bmod 9$ and $x \equiv 5 \bmod 11$. Such x necessarily have the form $x = 4 + 9y$, with $y \in \mathbb{Z}$. Moreover we demand $x = 4 + 9y \equiv 5 \bmod 11$, i.e., $9y \equiv 1 \bmod 11$. This means precisely that $y \bmod 11$ is the inverse of $9 \bmod 11$ in $(\mathbb{Z}/11\mathbb{Z})^\times$, so $y \bmod 11 = 5 \bmod 11$. So $y = 5 + 11z$, hence we conclude that $x = 4 + 9(5 + 11z) = 49 + 99z$ with $z \in \mathbb{Z}$ arbitrary. Put differently: the set of solutions equals the residue class of 49 modulo 99. The only integer between 1 and 99 with remainder 4 upon division by 9 and remainder 5 upon division by 11, is therefore 49.

II.3.6 Remark. Using mathematical induction with respect to n one may generalize the Chinese remainder theorem as follows. Suppose $N_1, \ldots, N_n$ are positive integers and $\gcd(N_i, N_j) = 1$ for all pairs i, j with $1 \le i < j \le n$. Then

$$\mathbb{Z}/N_1 \ldots N_n\mathbb{Z} \longrightarrow \mathbb{Z}/N_1\mathbb{Z} \times \mathbb{Z}/N_2\mathbb{Z} \times \ldots \times \mathbb{Z}/N_n\mathbb{Z}$$

given by $a \bmod N_1 \ldots N_n \mapsto (a \bmod N_1, \ldots, a \bmod N_n)$ defines a well-defined map which is bijective. The same holds if one restricts the map to units.

II.3.7 Example. The integers $7, 11$, and 13 are pairwise coprime, with product $7 \cdot 11 \cdot 13 = 1001$. For every triple of integers $a, b, c \in \mathbb{Z}$ there exists a unique $x \in \mathbb{Z}$ with $0 \le x \le 1000$ and $x \equiv a \bmod 7$ and $x \equiv b \bmod 11$ and $x \equiv c \bmod 13$. Try for yourself to find this x for certain triples a, b, c. ∎

II.3.8 Corollary. *Euler's totient function has the property $\varphi(NM) = \varphi(N)\cdot\varphi(M)$ for all positive coprime integers N, M.*

Proof. By definition $\varphi(n)$ equals the number of elements of $(\mathbb{Z}/n\mathbb{Z})^\times$. Hence the assertion is a direct consequence of Theorem II.3.4, using the fact that for finite sets V, W the number of elements of $V \times W$ equals the product of the numbers of elements of V and of W. ∎

Corollary II.3.8 states that Euler's totient function is a *multiplicative* function. Using this corollary we will derive a formula for $\varphi(n)$ in terms of the prime factorization of n. This uses the next result.

II.3.9 Lemma. *For p prime and k an integer ≥ 1 we have*

$$\varphi(p^k) = (p-1)p^{k-1} = p^k - p^{k-1}.$$

Proof. We know that $\varphi(p^k)$ equals the number of integers a with $0 \leq a \leq p^k - 1$ and $\gcd(a, p^k) = 1$. An integer is *not* coprime to p^k, precisely when it is divisible by p. The integers in the interval $0 \leq a \leq p^k - 1$ interval which are divisible by p are $0\cdot p, 1\cdot p, \ldots, m\cdot p$, with m the largest integer smaller than p^{k-1}. The interval therefore contains p^{k-1} integers divisible by p, but since it contains precisely p^k integers in total, we obtain $\varphi(p^k) = p^k - p^{k-1}$. ∎

Incidentally, Lemma II.3.9 also shows that the coprimality condition in Corollary II.3.8 is necessary.

II.3.10 Theorem. *For $n \geq 2$, Euler's totient function can be computed using the formula*

$$\varphi(n) = \prod_{p|n}(p-1)p^{v_p(n)-1} = n\prod_{p|n}(1-\frac{1}{p})$$

where the product is taken over the prime divisors of n.

Proof. The second equality follows using $n = \prod_{p|n} p^{v_p(n)}$.

For the first equality we will use mathematical induction with respect to N, showing that the formula holds for every n with $2 \leq n \leq N$. For $N = 2$ this is clear. Assume that the formula holds for $N \geq 2$. Then we only need to show the formula for $n = N+1$ to finish the case $N+1$. If this n is a power of a prime, then we are done by Lemma II.3.9. If not, write $n = p^{v_p(n)}\cdot n'$, with $2 \leq n' \leq N$. Then $\gcd(p^{v_p(n)}, n') = 1$, so by Corollary II.3.8 we have $\varphi(n) = \varphi(p^{v_p(n)})\varphi(n')$. Applying the induction hypothesis to n' then implies the formula for n (using that $v_q(n) = v_q(n')$ for every prime $q \neq p$). ∎

II.3.11 Example. $\varphi(1000000) = 2^5\cdot 4\cdot 5^5 = 400000$. So 400000 positive odd integers below one million exist with last decimal digit different from 5. ∎

II.4 Exercises

1. Prove that for every odd $n \in \mathbb{Z}$ the congruence $n^2 \equiv 1 \bmod 8$ holds, and for every odd prime $p \neq 3$ we even have $p^2 \equiv 1 \bmod 24$.
2. Let $n = \sum a_i 10^i$ be an integer (with all $a_i \in \mathbb{Z}$).
   (a) Show: for $p = 2$ and for $p = 5$ we have $p \mid n$ if and only if $p \mid a_0$.
   (b) Show: for $m = 3$ and for $m = 9$ one has $m \mid n$ if and only if $m \mid \sum a_i$.
   (c) Prove that $11 \mid n$ if and only if $11 \mid \sum (-1)^i a_i$.
3. Determine the inverse of $\overline{100}$ in $(\mathbb{Z}/257\mathbb{Z})^\times$.
4. Show that $2^{341} \equiv 2 \bmod 341$. Is 341 prime? Find an integer between 0 and 341 that is congruent to $3^{341} \bmod 341$.
5. Show that every $n \in \mathbb{Z}$ satisfies $n^{13} \equiv n \bmod 2730$.
6. Find the remainder upon dividing $(177 + 10^{15})^{116}$ by $1003 = 17 \times 59$.
7. Find all integers which leave a remainder 3 upon division by 7, remainder 6 upon division by 11, and remainder 1 upon division by 13.
8. (a) Determine for $n = 4$ the residue class $(n-1)! \bmod 4$.
   (b) Show that if $n > 4$ is not prime, then $(n-1)! \equiv 0 \bmod n$.
   (c) Now suppose $n = p$ is prime. Find all residue classes $a \bmod p \in (\mathbb{Z}/p\mathbb{Z})^\times$ satisfying $(a \bmod p)^{-1} = a \bmod p$.
   (d) Show for $n = p$ prime that $(n-1)! \equiv -1 \bmod n$.
   (e) Show that $n \geq 2$ is prime if and only if $(n-1)! \equiv -1 \bmod n$. Is this a practical way to test primality?
9. (a) Prove that a Carmichael number is not divisible by any square > 1.
   (b) Prove that if $n = p_1 \cdot \ldots \cdot p_t$ is a product of $t > 1$ distinct primes, and $p_i - 1 \mid n - 1$ for every i, then n is a Carmichael number.
10. Find all positive integers n satisfying $\varphi(n) = 24$. Answer the analogous question for $\varphi(n) = 14$.
11. Let $N, a \in \mathbb{Z}$ with $N > 0$.
    (a) Prove that $\overline{a} \in (\mathbb{Z}/N\mathbb{Z})^\times$ if and only if $\overline{-a} \in (\mathbb{Z}/N\mathbb{Z})^\times$.
    (b) Which residue classes $a \bmod N$ satisfy $a \bmod N = -a \bmod N$? (Distinguish the cases N even and N odd.)
    (c) Show that $\varphi(N)$ is even for all $N \geq 3$.
12. Show that if $p_1, \ldots, p_t$ are the smallest t primes, and $n_j = p_1 \cdots p_t - p_1 \cdots p_t / p_j$, then $\varphi(n_j) = \varphi(n_k)$ for $1 \leq j, k \leq t$. Conclude from this that for fixed m, the equation $\varphi(x) = m$ may have arbitrary many solutions.
13. Find all n such that $\varphi(n) \mid n$.

III GROUPS AND HOMOMORPHISMS

If we examine the properties of the integers $\mathbb{Z}$, equipped with the operation 'addition', we find that a number of them also hold for $\mathbb{Z}/N\mathbb{Z}$ with respect to addition, as well as for $(\mathbb{Z}/N\mathbb{Z})^\times$ with respect to multiplication. Below we will see many more sets equipped with an operation, all sharing the same properties. It is typical for (abstract) algebra to capture such a phenomenon in a definition. Instead of dealing with all separate cases one by one, this makes it possible to prove results at once for all examples satisfying the definition. We saw a similar situation in linear algebra: after abstractly introducing the notion 'vector space over $\mathbb{R}$', one deduces a range of properties not only for well known spaces such as $\mathbb{R}^2$ and $\mathbb{R}^3$, but also for planes and hyperplanes containing the origin in $\mathbb{R}^n$, for function spaces, spaces of polynomials, spaces of sequences, matrices, et cetera.

In this chapter we introduce the notion of a group. As the name of the course suggest, this is the central notion of this course. The first abstract definition of this concept was formulated by the German mathematician W.F.A. von Dyck (1856–1934). Algebra courses starting from abstract definitions of this kind were started in Göttingen around 1920, notably by the famous female mathematician Emmy Noether (1882–1935). A young student from Amsterdam, B.L. van der Waerden, attended her courses. He extended his algebra knowledge with the help of Emil Artin (1898–1962) in Hamburg. In 1928, only 25 years old, Van der Waerden was appointed mathematics professor in Groningen where he wrote what is probably the most influential textbook on abstract algebra to date. It appeared in 1930 and completely adopts the abstract definition/theorem/proof style. The book made Van der Waerden, who died in 1996, world famous. Due to Noether's and Artin's lectures and Van der Waerden's recording of this, abstract algebra is still taught all over the world essentially exclusively in this style.

III.1 Groups

III.1.1 Definition. A *group* is a triple $(G, \cdot, e)$ where G is a set, $e \in G$, and $\cdot$ is a map from $G \times G$ to G, which we write as $(x, y) \mapsto x \cdot y$, satisfying

G1 (associativity) For all $x, y, z \in G$ we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
G2 (unit element) For all $x \in G$ we have $e \cdot x = x = x \cdot e$.
G3 (inverses) For all $x \in G$ a $y \in G$ exists such that $x \cdot y = e = y \cdot x$.

A group $(G, \cdot, e)$ is called *commutative* or (in honour of the Norwegian mathematician Niels Henrik Abel, 1802–1829) *abelian*, if moreover, the following holds:

G4 For all $x, y \in G$ we have $x \cdot y = y \cdot x$.

The *order* of the group $(G,\cdot,e)$ is the number $\#G$ of elements of G. We call a group *finite* if it has finite order.

III.1.2 Remark. Instead of $(G,\cdot,e)$ we often simply write G, if the map $G\times G\to G$ and the element $e\in G$ are clear from the context. We abbreviate $x\cdot y$ as xy. The map $\cdot$ is called the *group law* on G. Alternatively, it is often called *multiplication* of G, although it does not have to be an actual 'multiplication' in the usual sense, see Example III.1.3. Accordingly, depending on the context, other notations are used for the group law, such as $x\circ y$ or $x*y$ or $x\times y$ or $x+y$. The latter notation is reserved for abelian groups.

III.1.3 Example. $(\mathbb{Z},+,0)$ is a group, as well as $(\mathbb{Z}/N\mathbb{Z},+,\overline{0})$ and $((\mathbb{Z}/N\mathbb{Z})^\times,\cdot,\overline{1})$. These are examples of commutative groups. Another example is $(\mathbb{R}^\times,\cdot,1)$, where $\mathbb{R}^\times=\mathbb{R}\setminus\{0\}$. ∎

III.1.4 Example. We already know many more commutative groups: if $G=V$ is a vector space (over $\mathbb{R}$) with vector addition '+' and zero vector $0\in V$, then $(V,+,0)$ is an abelian group. In other words, a vector space can be viewed as an abelian group by forgetting the scalar multiplication.

As a side note, we remark here that one can also consider vector spaces over the complex numbers, or the rational numbers or, more generally, any *field*, an important algebraic structure which will be introduced in the course 'Algebraic Structures'. The only difference is that one has to consider scalar multiplication by elements of such a field rather than $\mathbb{R}$; but otherwise, these vector spaces all share the same defining properties. In particular, they form an abelian group under addition. ∎

III.1.5 Example. The set of invertible $n\times n$ matrices with coefficients in $F=\mathbb{R}$ or $F=\mathbb{C}$ (or more generally: in a *field* F) becomes a group whose group law is the multiplication of matrices, and whose unit is the unit matrix. This group is denoted by $\mathrm{GL}_n(F)$. For $n\geq 2$ this group is *not* commutative, because (for example)

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0\\ 1 & 1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & \cdots & \vdots\\ \vdots & & & & \vdots\\ 0 & & \cdots & & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & \cdots & \vdots\\ \vdots & & & & \vdots\\ 0 & & \cdots & & 1 \end{pmatrix}$$
$$\neq$$
$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & \cdots & \vdots\\ \vdots & & & & \vdots\\ 0 & & \cdots & & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0\\ 1 & 1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & \cdots & \vdots\\ \vdots & & & & \vdots\\ 0 & & \cdots & & 1 \end{pmatrix}.$$

∎

Given elements $a_1,a_2,\ldots,a_n$ in a group G, their product $a_1a_2\ldots a_n$ is defined inductively with respect to n. For $n=2$ it is just the product with respect to the group law. If the product has already been defined for $n-1\geq 2$, then

$$a_1a_2\ldots a_n := (a_1\ldots a_{n-1})\cdot a_n.$$

Using associativity and induction with respect to n, it turns out that

$$(a_1a_2\ldots a_k)\cdot(a_{k+1}\ldots a_n)=(a_1a_2\ldots a_n).$$

This product of n factors is usually abbreviated as $\prod_{i=1}^n a_i$. If G is an abelian group whose group law is denoted '$+$', then this is abbreviated as $\sum_{i=1}^n a_i$ instead. If all a_i are equal, say $a_i = a$, then we write $a^n = \prod_{i=1}^n a_i$ (or $na = \sum_{i=1}^n a_i$ if G is abelian with group law '$+$'). The property mentioned above, writing $\ell = n - k$, translates into $a^k \cdot a^\ell = a^{k+\ell}$. Note that in general exponentiation in a group might not quite as nicely as exponentiation with e.g. integers, unless the group is abelian. For example the property $(AB)^2 = A^2B^2$ does *not* hold in general for invertible matrices $A, B \in \mathrm{GL}_n(\mathbb{R})$ (find an explicit example yourself).

We now present some elementary properties of groups.

III.1.6 Theorem. *Let $(G, \cdot, e)$ be a group.*

1. *If $e' \in G$ satisfies $e'x = x$ or $xe' = x$ for some $x \in G$, then $e' = e$.*
2. *For every $x \in G$ there is precisely one $y \in G$ with $xy = e = yx$.*
3. *For any fixed $a \in G$, the map $\lambda_a : G \to G$; $x \mapsto ax$ is a bijection from G to itself. Similarly, $\rho_a : G \to G$; $x \mapsto xa$ is a bijection.*

Proof. 1: If $x \in G$ satisfies $e'x = x$, then multiplying on the right by some $y \in G$ with $xy = e$ (such a y exists because of group property G3) shows $e' = e'e = e$; here the first equality follows from group property G2. The case $xe' = x$ is dealt with in a similar way, by multiplying on the left with $y \in G$ such that $yx = e$.
2: If $xy = e = yx$ and $xz = e = zx$ for some $y, z \in G$, then $z = ze = z(xy) = (zx)y = ey = y$, so $y = z$.
3: Let $a \in G$ and let $b \in G$ with $ba = e$. We first show that λ_a is injective. Suppose that $ax = ay$, then we find

$$x = ex = (ba)x = b(ax) = b(ay) = (ba)y = ey = y,$$

so $x = y$.

To see that λ_a surjective as well, let $z \in G$ and define $x := bz$. Then x is mapped to

$$ax = a(bz) = (ab)z = ez = z,$$

so z is in the image. The map λ_a is therefore both injective and surjective, hence bijective. The case of ρ_a is completely analogous. ∎

III.1.7 Definition. Let $(G, \cdot, e)$ be a group and $x \in G$. The element $y \in G$ such that $xy = e = yx$ is called the *inverse* of x in G. It is denoted by x^{-1}, by Theorem III.1.6 x^{-1} is unique and hence well-defined.

In case of an abelian group G with group law denoted as $+$, this inverse element is called the *opposite* of x in G, and it is denoted as $-x$.

III.1.8 Remark. To check that some element y in a group is the inverse of an element x, it suffices to verify that $xy = e$ (i.e. the other condition $yx = e$ then follows automatically). Namely, Theorem III.1.6(3.) implies that only one element in the group has this property, and by definition x^{-1} has this property. Similarly, it is enough to check that $yx = e$.

III.1.9 Corollary. Let G be a group and let $a, a_1, a_2, \ldots, a_n \in G$. Then we have

1. $(a^{-1})^{-1} = a$.
2. $(a_1 \cdots a_n)^{-1} = a_n^{-1} \cdot a_{n-1}^{-1} \cdots a_1^{-1}$. (When taking an inverse, the order reverses!)
3. $(a^n)^{-1} = (a^{-1})^n$.

Proof. 1: Note that $aa^{-1} = e = a^{-1}a$, which says that a is the inverse of a^{-1}, i.e., $(a^{-1})^{-1} = a$.

2: We show this by induction with respect to n. For $n = 1$ the statement holds. If $n \geq 2$ and we assume $(a_1 \cdots a_{n-1})^{-1} = a_{n-1}^{-1} \cdots a_1^{-1}$, then

$$(a_n^{-1} \cdot a_{n-1}^{-1} \cdots a_1^{-1}) \cdot (a_1 \cdots a_n) = a_n^{-1}((a_{n-1}^{-1} \cdots a_1^{-1}) \cdot (a_1 \cdots a_{n-1}))a_n = a_n^{-1} e a_n = e$$

and similarly $(a_1 \cdots a_n) \cdot (a_n^{-1} \cdot a_{n-1}^{-1} \cdots a_1^{-1}) = e$. This finishes the proof.
3: This is statement 2. with all a_i equal to a. ∎

If x is an element of a group G and $n \in \mathbb{Z}$, then we already defined x^n for positive n. For $n = 0$ we define $x^0 = e$, and for negative n we define $x^n = (x^{-1})^{-n}$. Using Corollary III.1.9 we see that $x^n \cdot x^{-n} = e$, so x^{-n} is the inverse of x^n. Moreover, using mathematical induction with respect to $|n|$ it is not hard to verify that $x^{n+m} = x^n \cdot x^m$ for $n, m \in \mathbb{Z}$.

III.1.10 Definition. The multiplication table
One can describe a group G consisting of only finitely many elements by means of a table which contains all results of multiplying pairs of elements of G. We represent this in a matrix $(a_{i,j})$. Position $a_{1,1}$ remains empty, or we could write the name of the group here. In the remainder of the first row we write the elements of G, and the same in the first column. In position $a_{i,j}$ (with $i, j \geq 2$) we put the product $a_{i,1} \cdot a_{1,j}$. (The product with an element from column 1 on the left, and an element from row 1 on the right! the order obviously makes a difference in the case of non-abelian groups.)

III.1.11 Example. Here is the multiplication table of $\mathbb{Z}/3\mathbb{Z}$:

$\mathbb{Z}/3\mathbb{Z}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{0}$	$\overline{1}$

▬∎

The fact that left multiplication by a fixed element is bijective, precisely means that in every row, all elements of the group appear exactly once (after the first position). In the same way right multiplication by a fixed element is bijective, and this says that in every column, from the second position onward, all elements occur exactly once.

The multiplication table can be used to check whether a finite group is abelian. This is the case if and only if $a_{i,j} = a_{j,i}$ for all i, j. In other words, the group is abelian if and only if the matrix is symmetric.

III.1.12 Example. Here is the multiplication table of a non-abelian group consisting of 6 elements. It turns out that all groups of order less than 6 are abelian, so in a sense this is the simplest non-abelian group.

G	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	f	d	c	b
b	b	d	e	f	a	c
c	c	f	d	e	b	a
d	d	b	c	a	f	e
f	f	c	a	b	e	d

▬∎

III.2 Subgroups

We now define the notion of a subgroup of a group, which is completely analogous to the notion of a linear subspace of a vector space.

III.2.1 Definition. Let $G = (G, \cdot, e)$ be a group. A group H is called a *subgroup* of G if H is a subset of G, and the unit element and the group law of H and G are the same. In this case we write $H \leqslant G$. We call H a *proper* subgroup if H is a proper subset of G.

In other words, a subset H of G forms a subgroup of G if it is itself a group under the same group law and with the same unit element as G.

III.2.2 Example. Every group $G = (G, \cdot, e)$ has the *trivial subgroup* $\{e\}$.

In $(\mathbb{Z}, +, 0)$ the set of all even integers is a subgroup.

The set $\mathbb{Z}_{\geqslant 0}$ consisting of all non-negative integers is *not* a subgroup of $\mathbb{Z}$. Namely, not every element x in the subset satisfies property G3 with respect to the group law on $\mathbb{Z}$ (addition).

In the group $\mathrm{GL}_n(\mathbb{R})$ the matrices with determinant 1 form a subgroup. This follows from the formula $\det(AB) = \det(A)\det(B)$ for $n \times n$-matrices A, B. This subgroup is denoted by $\mathrm{SL}_n(\mathbb{R})$. The same conclusion holds if we replace $\mathbb{R}$ by $\mathbb{Q}$ or by $\mathbb{C}$ or more generally, by an arbitrary field F. In this case, the subgroup of matrices with determinant 1 is denoted $\mathrm{SL}_n(F)$. ▬

To test whether a given subset of a group is a subgroup, one can use the following criterion.

III.2.3 Theorem. (Subgroup criterion) *Let $(G, \cdot, e)$ be a group and $H \subset G$. Then H forms a subgroup of G if and only if*

H1 $e \in H$.
H2 *For all $x, y \in H$ also $x \cdot y \in H$.*
H3 *For all $x \in H$ also $x^{-1} \in H$.*

Proof. If H forms a subgroup of $(G, \cdot, e)$, then by definition $(H, \cdot, e)$ is a group. This implies the properties H1, H2, and H3.

Vice versa, suppose that a subset H has properties H1, H2, and H3. We have to show that $(H, \cdot, e)$ is a group. H1 says that indeed $e \in H$, and H2 says that the restriction to H of the group law on G gives a map $H \times H \to H$. The triple $(H, \cdot, e)$ satisfies G1 and G2, since these properties hold for all of G hence also for the subset H. Finally, the triple $(H, \cdot, e)$ satisfies G3 because of H3. ∎

III.2.4 Remark. If H and H' are subgroups of a group G, then $H \cap H'$ is a subgroup of both H and H'. In particular, if in addition $H' \subset H$, then H' is automatically a subgroup of H.

III.2.5 Example. If G is a group and $x \in G$, then one easily checks that the set of all powers of x (positive as well as negative powers, and also $x^0 = e$) satisfies H1, H2, and H3. Hence forms a subgroup, the *subgroup generated by x*, which we denote by $\langle x \rangle$; some texts also use the notation $x^{\mathbb{Z}}$. ▬

III.2.6 Example. In $(\mathbb{Z}/24\mathbb{Z})^\times$ we find various subgroups $\langle x \bmod 24 \rangle = \langle \overline{x} \rangle$, namely $\langle \overline{1} \rangle$ consisting of only one element, and $\langle \overline{5} \rangle, \langle \overline{7} \rangle, \langle \overline{11} \rangle, \langle \overline{13} \rangle, \langle \overline{17} \rangle, \langle \overline{19} \rangle, \langle \overline{23} \rangle$ each having precisely two elements.

In fact, $(\mathbb{Z}/24\mathbb{Z})^\times$ contains even more subgroups; for example also $\{\overline{\pm 1}, \overline{\pm x}\}$ for $x = 5, 7, 11$, each consisting of 4 elements. ▬

III.2.7 Example. We will now describe all subgroups of $(\mathbb{Z}, +, 0)$. To start with, the trivial subgroup $\{0\} = 0\mathbb{Z}$ is a subgroup. If H is a subgroup and $H \neq 0\mathbb{Z}$, then H contains an element $x \neq 0$. Since H is a group with respect to the usual addition, we also have $-x \in H$. So we may conclude that H contains at least one positive integer, and we denote the smallest positive integer contained in H by a.

Now we claim that $H = \langle a \rangle = a\mathbb{Z}$. Indeed, the second equality is simply the definition of $\langle a \rangle$ as given in Example III.2.5. To show that $H \supset \langle a \rangle$, one needs to verify that we have $an \in H$ for every $n \in \mathbb{Z}$. This can be done using mathematical induction with respect to $|n|$, using the properties H2 and H3 of H (work out the details yourself!). Vice versa, one has to show $H \subset a\mathbb{Z}$. Take an arbitrary $b \in H$ and set $d := \gcd(a, b)$. Then there exist $x, y \in \mathbb{Z}$ with $d = ax + by$. Now $ax \in H$ (this argument is explained above) and since $b \in H$, we also get $by \in H$. Property H2 therefore implies $d = ax + by \in H$. We have $1 \leq d \leq a$, so because a is by definition the smallest positive integer in H, it follows that $d = a$. This implies $a = d|b$, i.e., $b \in a\mathbb{Z}$. Having verified both inclusions we conclude $H = \langle a \rangle$.

An arbitrary subgroup of $\mathbb{Z}$ is therefore of the form $a\mathbb{Z}$ for some $a \in \mathbb{Z}_{\geqslant 0}$. Vice versa, any subset of $\mathbb{Z}$ with the indicated form is a subgroup. So we have described all subgroups of $\mathbb{Z}$. ▬■

In the next theorem we present an important property of subgroups of *finite* groups. The counting argument used in the proof deserves special attention: we will encounter this technique more often later on.

III.2.8 Theorem. (Theorem of Lagrange; Joseph Louis Lagrange, French mathematician, 1736–1813) *If H is a subgroup of a finite group G, then the order of H is a divisor of the order of G.*

Proof. For $x \in G$ consider the subset $xH = \{xy \mid y \in H\}$ of G. The union of all subsets of this form is all of G, since an arbitrary element $x \in G$ is an element of xH, because $e \in H$ and $x = xe$.

We now claim that all of these subsets have the same number of elements, i.e., $\#xH = \#yH$ for all $x, y \in G$. This follows from the fact that the map $f : xH \to yH$ given by $f(z) = yx^{-1}z$ is a bijection between xH and yH.

If $z \in xH$, then $z = xh$ for some $h \in H$, so $f(z) = yx^{-1}z = yx^{-1}xh = yh \in yH$. Therefore this map sends xH to yH. The map is bijective, since a short calculation shows that $g : yH \to xH$ given by $g(z) = xy^{-1}z$ is its inverse. The existence of a bijection between two finite sets means that these sets have the same number of elements.

We will now show that if $xH \cap yH \neq \emptyset$, then the two sets are equal: $xH = yH$. Namely, suppose $z \in xH \cap yH$. Then $z \in xH$, so we may write $z = xh_1$ for some $h_1 \in H$. Similarly $z = yh_2$ for an $h_2 \in H$. Now $xh_1 = yh_2$, and multiplying both sides on the right by h_1^{-1} or by h_2^{-1} shows $x = yh_2h_1^{-1}$ and $y = xh_1h_2^{-1}$. Therefore, we have written an arbitrary $xh \in xH$ as $xh = yh_2h_1^{-1}h = y(h_2h_1^{-1}h) \in yH$ and similarly an arbitrary $yh \in yH$ as $yh = xh_1h_2^{-1}h \in xH$. This shows $xH = yH$.

We have written G as a union of pairwise disjoint subsets, all having the same number of elements. As a consequence $\#G$ equals the product of the number of such subsets and the cardinality $\#xH = \#eH = \#H$ of the subsets. This implies $\#H \mid \#G$, which we wanted to prove. ■

In order to use this result in the case of subgroups of the form $\langle x \rangle$ we first give the following definition.

III.2.9 Definition. Let x be an element of a group G. Then we define the *order* of x, notation $\operatorname{ord}(x)$, as follows. If an integer $m > 0$ exists with $x^m = e$, then $\operatorname{ord}(x)$ is defined to be the smallest such m. Otherwise, we set $\operatorname{ord}(x) := \infty$.

III.2.10 Example. For every group $(G,\cdot,e)$ we have $\mathrm{ord}(e)=1$. Moreover, if some $x\in G$ satisfies $\mathrm{ord}(x)=1$, then $x=e$, because $\mathrm{ord}(x)=1$ implies $x=x^1=e$.

In $(\mathbb{Z}/5\mathbb{Z})^\times$ we have $\mathrm{ord}(\overline{1})=1$, $\mathrm{ord}(\overline{4})=2$ and $\mathrm{ord}(\overline{2})=\mathrm{ord}(\overline{3})=4$. ▬

III.2.11 Theorem. *Let G be a group and an element $x\in G$. Then the following statements hold true:*

1. $\mathrm{ord}(x)=\mathrm{ord}(x^{-1})$.
2. *If* $\mathrm{ord}(x)<\infty$, *then* $\langle x\rangle=\{x,x^2,\dots,x^{ord(x)}=e\}$.
3. $\mathrm{ord}(x)=\#\langle x\rangle$, *i.e. the order of the subgroup generated by x is the order of x.*
4. *If* $\#G<\infty$, *then also* $\mathrm{ord}(x)<\infty$ *and moreover* $\mathrm{ord}(x)\mid\#G$.
5. *If* $x^n=e$, *then* $\mathrm{ord}(x)\mid n$.

Proof. 1: If $x^m=e$, then $(x^{-1})^m=x^{-m}=(x^m)^{-1}=e$. Applying the above for both x and its inverse, we see that the set of integers m with $x^m=e$ equals the set of integers n such that $(x^{-1})^n=e$. (Note that this set may be empty!) In particular it follows that $\mathrm{ord}(x)=\mathrm{ord}(x^{-1})$.
2: Put $d=\mathrm{ord}(x)$. For $m\in\mathbb{Z}$ write $m=qd+r$ with $0\le r<d$. Then $x^m=(x^d)^q\cdot x^r=x^r$. So $\langle x\rangle\subset\{e,x,\dots,x^{d-1}\}$, which implies the equality.
3: The assertion obviously holds in case $\mathrm{ord}(x)=\infty$ so we will from now on assume that the order of x is finite. In this case 2. implies the result, provided we show that the elements of $\{e,x,\dots,x^{\mathrm{ord}(x)-1}\}$ are pairwise distinct. Suppose that $x^m=x^n$ with $0\le m\le n<\mathrm{ord}(x)$. Multiplying by the inverse of x^m yields $e=x^{n-m}$, where $0\le n-m<\mathrm{ord}(x)$. Since $\mathrm{ord}(x)$ is defined to be the least positive d with $x^d=e$, we have $n-m=0$. So $x^m=x^n$ for nonnegative $n,m<\mathrm{ord}(x)$ is only possible when $n=m$. This shows 3).
4: $\langle x\rangle$ is a subgroup of G, and since G is finite, so is $\langle x\rangle$. Now 3. and Theorem III.2.8 imply $\mathrm{ord}(x)=\#\langle x\rangle$ is finite and $\mathrm{ord}(x)=\#\langle x\rangle\mid\#G$.
5: $x^n=e$ implies $\mathrm{ord}(x)<\infty$. Put $d=\gcd(n,\mathrm{ord}(x))$. Then integers k,ℓ exist with $nk+\mathrm{ord}(x)\ell=d$. We have $x^d=(x^n)^k(x^{\mathrm{ord}(x)})^\ell=e$. Since $1\le d\le\mathrm{ord}(x)$, the definition of $\mathrm{ord}(x)$ implies $d=\mathrm{ord}(x)$. In particular, $\mathrm{ord}(x)=d=\gcd(n,\mathrm{ord}(x))\mid n$. ■

III.2.12 Example. By Theorem III.2.8 and Theorem III.2.11. In a finite group both the number of elements of any subgroup and the order of any element is a divisor of the number of elements of the group.

However, not every divisor of the number of elements of the group necessarily appears as the order of some element. For example, we already saw in Example III.2.6 that all elements of $(\mathbb{Z}/24\mathbb{Z})^\times$ except the unit element, have order 2. We will return to this issue later on. ▬

III.2.13 Definition. The product of groups
Given two groups $(G_1,\cdot,e_1)$ and $(G_2,*,e_2)$, the product set $G_1\times G_2$ can be given the structure of a group as follows. By definition, the elements of $G_1\times G_2$ are all ordered pairs (x_1,x_2) with $x_i\in G_i$. The unit element is the pair (e_1,e_2). The group law is given by $(x_1,x_2)\circ(y_1,y_2):=(x_1\cdot y_1,x_2*y_2)$. Check for yourself that indeed with these definitions $(G_1\times G_2,\circ,(e_1,e_2))$ is a group. We call it the *(direct) product* of the groups G_1 and G_2.

We define the product of more than two groups analogously.

III.2.14 Example. The groups $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ each have 8 elements, and three groups are commutative. Nevertheless in some sense they are very different: the first one contains 4 elements of order 8 while the other two have no such elements. The second group has 4 elements of order 4, whereas the last one only contains elements of order 1 and 2. Incidentally, there are also two "quite different" non-commutative groups with exactly 8 elements (see Exercise 17). ▬

III.3 Homomorphisms

After groups and subgroups we now discuss maps between groups. In courses on linear algebra the maps considered between vector spaces are the linear maps (or linear transformations), i.e. maps preserving the operations (scalar multiplication and addition) defined on vectors. We will follow a similar approach for groups.

This is a situation encountered frequently in mathematics. One studies not only mathematical objects having some structure (such as vector spaces or groups), but also those maps between them which preserve their structure.

III.3.1 Definition. Let $(G_1,\cdot,e_1)$ and $(G_2,*,e_2)$ be groups. A *homomorphism* from G_1 to G_2 is a map $f : G_1 \to G_2$ satisfying $f(x\cdot y) = f(x) * f(y)$ for all $x,y \in G_1$. An *isomorphism* from G_1 to G_2 is a bijective homomorphism.

We call G_1 and G_2 *isomorphic*, and write $G_1 \cong G_2$ if an isomorphism from G_1 to G_2 exists.

III.3.2 Example. 1. Let $\mathbb{R}_{>0}$ denote the positive real numbers. This is a group under the usual multiplication. The map $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ given by $x \mapsto e^x$ is an isomorphism from $(\mathbb{R},+,0)$ to $(\mathbb{R}_{>0},\cdot,1)$. Namely, the map exp is bijective, and $\exp(x+y) = e^{x+y} = e^x e^y = \exp(x)\exp(y)$. Hence the two groups are isomorphic.

2. $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$ is a homomorphism, since $\det(AB) = \det(A)\det(B)$. For $n \neq 1$ this is not an isomorphism, because for $n > 1$ one easily finds two distinct invertible matrices with equal determinant. However, det is clearly surjective.

3. If G is an arbitrary group and $x \in G$, then $f_x : \mathbb{Z} \to G$ given by $f(n) = x^n$ is a homomorphism, since $f_x(n+m) = x^{n+m} = x^n x^m = f_x(n) f_x(m)$.
In the special case $G = \mathbb{Z}/N\mathbb{Z}$ and $x = \overline{1}$ this map $f_{\overline{1}}$ is the "reduction modulo N": $n \mapsto n \bmod N$.

4. The map $a \bmod NM \mapsto a \bmod N$ used in the Chinese Remainder Theorem II.3.4 is a homomorphism from $(\mathbb{Z}/NM\mathbb{Z},+,0 \bmod NM)$ to $(\mathbb{Z}/N\mathbb{Z},+,0 \bmod N)$ and also from $((\mathbb{Z}/NM\mathbb{Z})^\times,\cdot,1 \bmod NM)$ to $((\mathbb{Z}/N\mathbb{Z})^\times,\cdot,1 \bmod N)$. In particular it follows that if $\gcd(N,M) = 1$, then

$$(\mathbb{Z}/NM\mathbb{Z},+,0 \bmod NM) \cong (\mathbb{Z}/N\mathbb{Z},+,0 \bmod N) \times (\mathbb{Z}/M\mathbb{Z},+,0 \bmod M)$$

and

$$((\mathbb{Z}/NM\mathbb{Z})^\times,\cdot,1 \bmod NM) \cong ((\mathbb{Z}/N\mathbb{Z})^\times,\cdot,1 \bmod N) \times ((\mathbb{Z}/M\mathbb{Z})^\times,\cdot,1 \bmod M).$$

∎

We present some basic properties of homomorphisms.

III.3.3 Theorem. *Given a homomorphism $f : (G_1,\cdot,e_1) \to (G_2,*,e_2)$, the following holds true:*

1. *$f(e_1) = e_2$.*
2. *If $x \in G_1$, then we have $f(x^{-1}) = (f(x))^{-1}$.*
3. *If f is an isomorphism, then so is the inverse of f.*
4. *If $g : (G_2,*,e_2) \to (G_3,\star,e_3)$ is a homomorphism as well, then so is the composition $g \circ f$.*

Proof. 1: Write $a = f(e_1)$. In G_2 we have $a * a = f(e_1) * f(e_1) = f(e_1 \cdot e_1) = f(e_1) = a$. Multiplying both sides by the inverse of a yields $a = e_2$.
2: Put $y = f(x^{-1})$. Then $y * f(x) = f(x^{-1}) * f(x) = f(x^{-1}\cdot x) = f(e_1) = e_2$. So y is the

inverse of $f(x)$, which is what we wanted to prove.
3: Denote the inverse of the map f by h and let $x, y \in G_2$. Then

$$f(h(x * y)) = x * y = f(h(x)) * f(h(y)) = f(h(x) \cdot h(y)).$$

Since f is bijective this implies $h(x * y) = h(x) \cdot h(y)$. So h is a homomorphism. Bijectivity of f implies that its inverse h is bijective as well. So h is an isomorphism.
4: $g \circ f(x \cdot y) = g(f(x \cdot y)) = g(f(x) * f(y) = g(f(x)) \star g(f(y)) = g \circ f(x) \star g \circ f(y)$ holds for $x, y \in G_1$. ■

An isomorphism of groups may be considered as a kind of name changer: the elements $x \in G_1$ obtain a new name $f(x) \in G_2$, and the group law is renamed as well. Yet, in a sense nothing has changed. For instance, two finite groups G_1 and G_2 are isomorphic if and only if we can get a multiplication table for G_2 from a multiplication table for G_1 by relabeling the elements. In particular the order of an element remains the same (in spite of the element obtaining a new name). In various cases this observation may be used to show that certain groups are *not* isomorphic (compare Example III.2.14).

Recall some notations concerning a function φ from a set S_1 to a set S_2: if $T_1 \subset S_1$, then the *image* of T_1, denoted by $\varphi(T_1)$, is defined as

$$\varphi(T_1) = \{y \in S_2 \mid x \in S_1 \text{ exists with } y = \varphi(x)\}.$$

Similarly for $T_2 \subset S_2$ the *preimage* of T_2, denoted by $\varphi^{-1}(T_2)$, is defined as

$$\varphi^{-1}(T_2) = \{x \in S_1 \mid \varphi(x) \in T_2\}.$$

In group theory, the special case where $\varphi = f$ is a homomorphism and the T_i are subgroups is of great importance.

III.3.4 Theorem. *Let $f : (G_1, \circ, e_1) \to (G_2, *, e_2)$ a homomorphism and let $H_i \leqslant G_i$ be subgroups for $i = 1, 2$. Then $f(H_1)$ is a subgroup of G_2, and $f^{-1}(H_2)$ is a subgroup of G_1.*

Proof. For both assertions it suffices to check the conditions H1, H2, and H3. H1: $e_2 \in f(H_1)$, since $e_1 \in H_1$ and $f(e_1) = e_2$. Moreover $e_1 \in f^{-1}(H_2)$, because $f(e_1) = e_2 \in H_2$.

Now condition H2: let $x, y \in f^{-1}(H_2)$. Then $f(x), f(y) \in H_2$, so because H_2 is a group, $f(x \cdot y) = f(x) * f(y) \in H_2$ as well. This means $x \cdot y \in f^{-1}(H_2)$. If $w, z \in f(H_1)$, then by definition $u, v \in H_1$ exist with $f(u) = w$ and $f(v) = z$. Now H_1 is a group, so $u \cdot v \in H_1$, and therefore $w * z = f(u) * f(v) = f(u \cdot v) \in f(H_1)$.

Finally H3: for $x \in f^{-1}(H_2)$ we know $f(x^{-1}) = (f(x))^{-1} \in H_2$, since $f(x) \in H_2$ and H_2 is a group. This implies $x^{-1} \in f^{-1}(H_2)$. If $z \in f(H_1)$, then write $z = f(v)$ with $v \in H_1$. Then $z^{-1} = (f(v))^{-1} = f(v^{-1}) \in f(H_1)$, since $v^{-1} \in H_1$. This proves the theorem. ■

III.3.5 Definition. If $f : (G_1, \cdot, e_1) \to (G_2, *, e_2)$ is a homomorphism, then the *kernel* of f, denoted by $\ker(f)$, is defined as

$$\ker(f) = \{x \in G_1 \mid f(x) = e_2\}.$$

III.3.6 Theorem. *Let $f : (G_1, \cdot, e_1) \to (G_2, *, e_2)$ be a homomorphism. Then*

1. *$\ker(f)$ is a subgroup of G_1;*
2. *f is injective if and only if $\ker(f) = \{e_1\}$.*

Proof. 1: This is a consequence of Theorem III.3.4, since $\{e_2\}$ is a subgroup of G_2, and $\ker(f) = f^{-1}(\{e_2\})$.
2: We know that $f(e_1) = e_2$ by part 1. of Theorem III.3.3. Now let $x \in G_1$ such that $f(x) = e_2 = f(e_1)$. If f is injective, this implies $x = e_1$, so $\ker(f) = \{e_1\}$. Vice versa, let $\ker(f) = \{e_1\}$. If $f(x) = f(y)$ for some $x, y \in G_1$, then

$$e_2 = f(x) * (f(x))^{-1} = f(y) * f(x^{-1}) = f(y \cdot x^{-1}).$$

So, $y \cdot x^{-1} \in \ker(f) = \{e_1\}$, i.e., $y \cdot x^{-1} = e_1$. This implies $x = y$, so f is injective. ∎

III.3.7 Remark. To test whether a homomorphism is injective, it is typically much easier to use Theorem III.3.6 than to use the definition of injectivity directly.

III.3.8 Remark. Every subgroup H of a group G can be written as $f(G')$ for some homomorphism f from some group G' to G. For this we simply take $G' = H$ and let f be the inclusion map $x \mapsto x$ from G' to G.

However it is *not* possible to realize every subgroup of any group G as the kernel of a homomorphism from G to another group G''. The problem is that a subgroup H which is of the form $H = \ker(f)$ for a homomorphism f, has the following special property: If $h \in H$, and if $x \in G$ is arbitrary, then also $x \cdot h \cdot x^{-1} \in H$. Kernels have this property since $f(x \cdot h \cdot x^{-1}) = f(x) * f(h) * (f(x))^{-1} = f(x) * e_2 * ((f(x))^{-1} = e_2$ for $h \in \ker(f)$, but not all subgroups have it.

You should check for yourself that, for example, the subgroup consisting of all upper triangular matrices in $\mathrm{GL}_2(\mathbb{R})$ does not satisfy this condition. We will see later (Chapter VII) that all subgroups which *do* have the given property, *can* in fact be realised as kernel of some homomorphism.

III.4 Exercises

1. Examine which of the following triples define a group:
   (a) $(\mathbb{N}, +, 0)$;
   (b) $(\mathbb{Q}_{>0}, \cdot, 1)$;
   (c) $(\mathbb{R}, \star, 1)$ with $x \star y = x + y - 1$;
   (d) $(\{x \in \mathbb{R} \mid -\pi/2 < x < \pi/2\}, \circ, 0)$ with $x \circ y = \arctan(\tan(x) + \tan(y))$;
   (e) $(\mathbb{Z}_{>0}, \bullet, 1)$ with $n \bullet m = n^m$.
2. Let V be a vector space over the real numbers. Show that the set $\mathrm{GL}(V)$ of all bijective linear maps from V to itself carries a group structure.
3. Prove that the subgroups of $\mathbb{Z}/N\mathbb{Z}$ are exactly given by $\langle a \bmod N \rangle$, for $a \mid N$.
4. Find all subgroups of $(\mathbb{Z}/24\mathbb{Z})^\times$.
5. Consider the 2×2 matrices that (with respect to the standard basis of $\mathbb{R}^2$) represent rotation around the origin over 120 degrees and reflection in the x-axis. Construct the smallest possible subgroup of $\mathrm{GL}_2(\mathbb{R})$ containing these two matrices. Is the resulting group abelian? Find the order of each element of this group.
6. Determine all subgroups of the group considered in Exercise 5. Check for each of them whether it can be written as the kernel of a homomorphism.
7. The *center* $\mathcal{Z}(G)$ of a group G is defined as $\mathcal{Z}(G) = \{x \in G \mid xy = yx \text{ for all } y \in G\}$.
   (a) Show that $\mathcal{Z}(G)$ is an abelian subgroup of G.
   (b) Determine $\mathcal{Z}(\mathrm{GL}_2(\mathbb{R}))$.
8. Find and prove a version of the subgroup criterion which combines H2 and H3 into one single condition.
9. Let G_1 and G_2 be finite groups. Show that $\mathrm{ord}(x, y) = \mathrm{lcm}(\mathrm{ord}(x), \mathrm{ord}(y))$ for all $(x, y) \in G_1 \times G_2$.
10. In $\mathbb{C}$ we define the subset $\mathbf{T} = \{a + bi \in \mathbb{C} \mid a^2 + b^2 = 1\}$. Denote $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.
    (a) Show that $(\mathbf{T}, \cdot, 1)$ is a subgroup of $(\mathbb{C}^\times, \cdot, 1)$.
    (b) Prove that $(\mathbb{C}^\times, \cdot, 1) \cong (\mathbf{T}, \cdot, 1) \times (\mathbb{R}_{>0}, \cdot, 1)$.

    The group $\mathbf{T}$ is often called the *circle group* for obvious reasons.
11. Suppose that G is a group and $f : G \to G$ is the map $x \mapsto x \cdot x$. Prove that f is a homomorphism if and only if G is abelian.
12. Suppose that $f : G_1 \to G_2$ is a surjective homomorphism of groups. Show that if G_1 is abelian, then so is G_2. Give an example where G_2 is abelian but G_1 is not.
13. (a) Show that an element x of a group G satisfies $x = x^{-1}$ if and only if $\mathrm{ord}(x) = 2$ or $\mathrm{ord}(x) = 1$.
    (b) Conclude that a finite group has an even number of elements if and only if it contains an element of order 2.
14. Show that the property 'being isomorphic' defines an equivalence relation.
15. Show that up to isomorphism exactly two groups consisting of 4 elements exist. In other words, find two non-isomorphic group of order 4 such that every group of order 4 is isomorphic to exactly one of them (this requires some puzzling; consider the possible orders of elements in such a group, and try to construct the possible multiplication tables).
16. Given a prime p and a group G with exactly p elements, take $x \in G$ with $x \neq e$. What is the order of x? Prove that $G \cong \mathbb{Z}/p\mathbb{Z}$. So, up to isomorphism only one group consisting of p elements exists, and every group of prime order is abelian.

17. In the group $\mathrm{GL}_2(\mathbb{C})$ consisting of all invertible 2×2 matrices with complex coefficients we consider two subgroups: H_1 is the minimal one containing both $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$; moreover H_2 is the minimal one containing both $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Verify that both subgroups are non-abelian, that each consists of 8 elements, and that H_1 and H_2 are not isomorphic (for example, count the elements of order 2 in both groups).
18. Let x be an element of a group G, and consider the homomorphism $f_x : \mathbb{Z} \to G$ given by $f(n) = x^n$. Find a relation between the order of x and the kernel of f_x.
19. Given a prime $p \neq 3$ and an $n \in \mathbb{Z}$ with $p \mid n^2 + n + 1$.
    (a) Verify that $n \bmod p \neq 1 \bmod p$ and that $n \bmod p \in (\mathbb{Z}/p\mathbb{Z})^\times$.
    (b) Show that $\mathrm{ord}(n \bmod p) = 3$ in the group $(\mathbb{Z}/p\mathbb{Z})^\times$.
    (c) Conclude that $p \equiv 1 \bmod 3$.
    (d) Prove that infinitely many primes $\equiv 1 \bmod 3$ exist. (Hint: if $p_1, \ldots, p_t$ are such primes, put $n = 3 \cdot p_1 \cdots p_t$ and consider prime divisors of $n^2 + n + 1$.)
    (e) (Compare Exercise 7 in Chapter I): Show that infinitely many primes p exist such that $p + 2$ is not prime.
20. Let $p \neq 2$ be prime and let $n \in \mathbb{Z}$ with $p \mid n^2 + 1$.
    (a) Verify (using an approach analogous to the one applied in Exercise 19) that $p \equiv 1 \bmod 4$.
    (b) Prove that infinitely many primes $\equiv 1 \bmod 4$ exist.

IV GROUPS OF PERMUTATIONS

The previous chapter contains an introduction to several abstract notions, such as groups, subgroups and homomorphims, but only few examples. The next two chapters are concerned with some important classes of groups, which will serve as examples throughout the course. We start groups consisting of all bijective maps from a set to itself. These are particularly useful in combinatorics.

IV.1 Bijections of a set

Let Σ be a non-empty set. Recall that a bijection from Σ to itself is a map $\sigma : \Sigma \to \Sigma$ which is injective as well as surjective. Such a σ has a unique inverse, say $\tau : \Sigma \to \Sigma$, satisfying $\sigma \circ \tau = \tau \circ \sigma = \mathrm{id}_\Sigma$. Here, $\circ$ is the composition of maps, and $\mathrm{id}_\Sigma : \Sigma \to \Sigma$ is the identity map given by $\mathrm{id}_\Sigma(x) = x$ for all $x \in \Sigma$. The composition of bijections is a bijection as well.

IV.1.1 Definition. For a non-empty set Σ one denotes by S_Σ the set of all bijections from Σ to itself. The *symmetric group* on the set Σ is defined as the group $(S_\Sigma, \circ, \mathrm{id}_\Sigma)$.

It is easily verified that the symmetric group is indeed a group.

IV.1.2 Example. In case Σ consists of only one element, the only bijection on Σ is the identity. In this case one obtains a group S_Σ consisting of only one element (the "trivial" group).

If Σ consists of two elements, precisely two bijections are possible: the one fixing both elements (this is id_Σ), and the one interchanging the two elements (let's call it τ). Then $\tau^2 = \tau \circ \tau = \mathrm{id}_\Sigma$. The group S_Σ is in this case isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

For Σ with $\#\Sigma > 2$ the group S_Σ is not commutative. Namely, take three distinct elements $x, y, z \in \Sigma$. Define two bijections $\sigma, \tau \in S_\Sigma$ as follows: σ interchanges x and y and it fixes all other elements of Σ. Similarly τ interchanges y and z and it fixes the remaining elements. This indeed defines two bijections, and $\sigma \circ \tau(x) = y$ whereas $\tau \circ \sigma(x) = z$. So $\sigma \circ \tau \neq \tau \circ \sigma$. In particular, S_Σ is not commutative. ∎

Given two "equally big" sets Σ and Σ' (more precisely: two sets with a bijection $f : \Sigma \xrightarrow{\sim} \Sigma'$), then intuitively it should be clear that the groups S_Σ and $S_{\Sigma'}$ are isomorphic. Indeed, the bijection f provides a way to give all elements of Σ a new name, and describing bijections in terms of either the old or the new names is essentially the same. Turning this argument into a formal proof yields the following:

IV.1.3 Theorem. *Suppose that $f : \Sigma \to \Sigma'$ is a bijection and $g : \Sigma' \to \Sigma$ is its inverse (so $f \circ g = \mathrm{id}_{\Sigma'}$ and $g \circ f = \mathrm{id}_{\Sigma}$). Then S_Σ and $S_{\Sigma'}$ are isomorphic; an explicit isomorphism $\varphi : S_\Sigma \to S_{\Sigma'}$ is given by $\varphi(\sigma) = f \circ \sigma \circ g$, with as inverse $\psi : S_{\Sigma'} \to S_\Sigma$ given by $\psi(\tau) = g \circ \tau \circ f$.*

Proof. This is a useful exercise in formal calculations with compositions of maps, and it tests understanding of a number of definitions. We leave it as an exercise. ■

Symmetric groups may look like rather special examples, but the following result shows that they are in fact very general.

IV.1.4 Theorem. (Cayley's theorem; Arthur Cayley, English mathematician, 1821–1895) *Every group G is isomorphic to a subgroup of S_G.*

Proof. For fixed $a \in G$ the map $\lambda_a : G \to G$ given by $\lambda_a(x) = ax$ is a bijection, see Theorem III.1.6. So $\lambda_a \in S_G$. We use this to define a map

$$\varphi : G \longrightarrow S_G$$

by $\varphi(a) = \lambda_a$. One easily verifies that φ is a homomorphism, i.e., for $a, b \in G$ one has $\varphi(ab) = \lambda_{ab} = \lambda_a \circ \lambda_b = \varphi(a) \circ \varphi(b)$.

The homomorphism φ is injective because any $a \in \ker(\varphi)$ satisfies by definition $\lambda_a = \mathrm{id}_G$, so $a = ae = \lambda_a(e) = \mathrm{id}_G(e) = e$. It follows that G is isomorphic to $\varphi(G)$, and the latter is indeed a subgroup of S_G. ■

IV.1.5 Remark. Historically, the concept of a symmetric group precedes the notion of an abstract group. Such groups were first studied in the work of the French mathematician Évariste Galois (1811 – 1832)[1] who studied polynomial equations in one variable by exploiting symmetries between their solutions or, in more modern terminology, by looking at the symmetric groups of the set of solutions. This has lead to a beautiful branch of mathematics called *Galois theory*, which in Groningen is taught in the course Advanced Algebraic Structures.

IV.2 Permutations on n integers

We now consider the special case of finite sets Σ. A bijection between two such sets exists precisely when they have the same number of elements. Hence Theorem IV.1.3 shows that when studying symmetry groups of finite sets, it suffices to consider the sets $S_{\{1,2,\ldots,n\}}$. Throughout this section, let $n \in \mathbb{Z}_{\geq 1}$.

IV.2.1 Definition. The *symmetric group on n integers*, denoted by S_n, is defined as the group $S_{\{1,2,\ldots,n\}}$. Elements of this group are called *permutations*. The group S_n is also called the *permutation group on n elements*.

We first record the following consequence of Cayley's Theorem IV.1.4 and of Theorem IV.1.3

IV.2.2 Corollary. A finite group G is isomorphic to a subgroup of $S_{\{1,\ldots,n\}}$.

A simple combinatorial argument yields the order of S_n.

IV.2.3 Theorem. *The group S_n consists of $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$ elements.*

[1] this is not a typo

Proof. An element of S_n is by definition a bijection of the set $\{1,\dots,n\}$. Such a bijection can be described as a sequence of length n, in which each integer $1,\dots,n$ appears exactly once. One readily verifies that precisely $n!$ such sequences exist. ∎

We now introduce some special permutations, which in a certain sense form building blocks for all permutations. These maps permute a subset of $\{1,\dots,n\}$ cyclically, and leave all other integers in $\{1,\dots,n\}$ fixed.

IV.2.4 Definition. A permutation $\sigma \in S_n$ is called a *cycle* of length k (or a *k-cycle*), if there exist k distinct integers $a_1,\dots,a_k \in \{1,\dots,n\}$ such that $\sigma(a_i) = a_{i+1}$ for $1 \le i < k$ and $\sigma(a_k) = a_1$ and $\sigma(x) = x$ for $x \notin \{a_1,\dots,a_k\}$. Such a permutation is denoted by $\sigma = (a_1\, a_2\, \dots\, a_k)$. A 2-cycle is also called a transposition.

If two cycles $(a_1\, a_2\, \dots\, a_k)$ and $(b_1\, b_2\, \dots\, b_\ell)$ satisfy $\{a_1,\dots,a_k\} \cap \{b_1,\dots,b_\ell\} = \varnothing$, they are called *disjoint*.

IV.2.5 Example. In S_n, where $n \geqslant 5$, we have $(1\,2\,3\,4\,5) = (2\,3\,4\,5\,1) = \dots = (5\,1\,2\,3\,4)$, since these 5-cycles send 5 to 1, and i to $i+1$ for $1 \le i < 5$, and they fix the integers ≥ 6 if $n \geqslant 6$. In general, the same reasoning shows for k-cycles that

$$(a_1\, a_2\, \dots\, a_k) = (a_2\, \dots\, a_k\, a_1) = \dots = (a_k\, a_1\, \dots\, a_{k-1}).$$

Two disjoint cycles commute, because if $(a_1\, a_2\, \dots\, a_k)$ and $(b_1\, b_2\, \dots\, b_\ell)$ are disjoint, then the first one only affects the integers $a_1,\dots,a_k$ and the second one only $b_1,\dots,b_\ell$. Hence it is irrelevant in which order these cycles are applied.

This is different for non-disjoint cycles: e.g., $(1\,2\,3)\circ(2\,3\,4) \neq (2\,3\,4)\circ(1\,2\,3)$, since the first composition maps 2 to 1 and the second one maps 2 to 4. (Note that we are composing functions, so the rightmost function is applied first!) ∎

IV.2.6 Theorem. *Every $\sigma \in S_n$ can be written as a product $\sigma = \sigma_1 \dots \sigma_r$, where the σ_i are pairwise disjoint cycles. Apart from the order of the σ_i, this presentation is unique.*

Proof. Existence can be shown using induction with respect to n, as follows. For $n = 1$ the assertion is clear, since in this case the only permutation is $\sigma = (1)$. Let $n > 1$ and assume the existence for all S_m with $m < n$. Let $\sigma \in S_n$, then $\{1, \sigma(1), \sigma^2(1), \dots\}$ is a subset of $\{1,\dots,n\}$, so k, ℓ exist with $k < \ell$ and $\sigma^k(1) = \sigma^\ell(1)$. One concludes $\sigma^{\ell-k}(1) = 1$, so there is a positive integer s satisfying $\sigma^s(1) = 1$. Denote the least such positive integer by q. The integers $1, \sigma(1), \dots, \sigma^{q-1}(1)$ are pairwise distinct by construction, and the effect of σ on these integers is given by the k-cycle $\sigma_1 = (1\, \sigma(1)\, \dots\, \sigma^{q-1}(1))$.

Now consider $T := \{1,\dots,n\} \setminus \{1, \sigma(1), \dots, \sigma^{q-1}(1)\}$. If T is the empty set, then $\sigma = \sigma_1$ and we are done. If T is nonempty, then σ acts as a permutation on it. The induction hypothesis implies that the restriction $\sigma|_T$ of σ to T can be written as a product of disjoint cycles $\sigma_2 \dots \sigma_r$. Considering these cycles as permutations on $\{1,\dots,n\}$ (which means that we extend them to elements of S_n by setting $\sigma_i(j) := j$ for $j \in \{1, \sigma(1), \dots, \sigma^{q-1}(1)\}$, we have $\sigma = \sigma_1 \dots \sigma_r$.

To show uniqueness, assume that some permutation allows two different presentations as product of disjoint cycles. Suppose $i \mapsto j$, then in both of the presentations exactly one cycle occurs containing $(\dots i\, j \dots)$. Now considering the image of j etc., shows that the presentations contain the same cycles, so they are equal. ∎

IV.2.7 Example. The argument above is in fact an algorithm. To illustrate this, suppose we want to write $(1\,2\,3\,4)(2\,3\,4\,5)(4\,5\,1)$ as a product of disjoint cycles. We see here a composition of maps. First, we determine its effect on 1. The rightmost permutation sends 1 to 4, and 4 is mapped by the middle permutation to 5. The

leftmost permutation fixes 5, so in total the image of 1 is 5. Next we find out what happens to 5. The rightmost sends 5 to 1; this 1 is fixed by the middle one and then sent to 2 by the remaining cycle. Continuing in this way we find that 4 is the image of 2, and 4 is mapped to 3, and 3 to 1. In this way we have found a 5-cycle, and since the initial permutations only involve the integers 1 to 5, we are done: $(1\,2\,3\,4)(2\,3\,4\,5)(4\,5\,1) = (1\,5\,2\,4\,3)$. ▬

Writing a permutation as a product of disjoint cycles helps us to determine the order of a permutation:

IV.2.8 Theorem. *Let $\sigma := (i_1\, i_2\, \ldots\, i_k) \in S_n$ be a k-cycle. Then we have*

1. $\sigma^{-1} = (i_k\, i_{k-1} \ldots i_1)$.
2. $\mathrm{ord}(\sigma) = k$.
3. *If $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint cycles, then $(\sigma_1 \ldots \sigma_r)^n = \sigma_1^n \ldots \sigma_r^n$ for all $n \in \mathbb{Z}$.*
4. *If, moreover, σ_i has length ℓ_i ($i = 1, \ldots, r$), then $\mathrm{ord}(\sigma_1 \ldots \sigma_r) = \mathrm{lcm}(\ell_1, \ldots, \ell_r)$.*

Proof. 1. This is immediate from the definition of a cycle.
2. For $0 < n < k$ the image of i_1 under $(i_1\, i_2\, \ldots\, i_k)^n$ equals i_{n+1}. Since $i_{n+1} \neq i_1$, this means the cycle has order $\geq k$. Now $(i_1\, i_2\, \ldots\, i_k)^k = (1)$, so the order equals k.
3. This is a consequence of the fact that disjoint cycles commute.
4. Using 3. and the uniqueness in Theorem IV.2.6 it follows that $(\sigma_1 \ldots \sigma_r)^n = (1)$ precisely when $\sigma_1^n = \ldots = \sigma_r^n = (1)$. By Theorem III.2.11 the latter holds if and only if n is a multiple of each of $\mathrm{ord}(\sigma_1), \ldots, \mathrm{ord}(\sigma_r)$. ■

IV.2.9 Example. The n-th power of a k-cycle, with $1 < n < k$, is not necessarily itself a k-cycle. As an example, $(1\,2\,3\,4)^2 = (1\,3)(2\,4)$. ▬

IV.2.10 Example. We determine which integers occur as order of some element in S_5. Note that we have

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

These are all presentations of 5 as a sum of positive integers. We find that a product of disjoint cycles in S_5 can be obtained in seven ways: a 5-cycle, or a 4-cycle (multiplied by a 1-cycle, which we leave out since it represents the identity map), etc. Theorem IV.2.8 implies that the orders of these products are $5, 4, 6, 3, 2$, and 1, respectively. For each of these numbers it is a relatively simple combinatorial problem, which we leave to the reader, to determine how many elements in S_5 have the given number as its order. ▬

IV.2.11 Theorem. *Every permutation $\sigma \in S_n$ can be written as a product of transpositions.*

Proof. We know that σ is a product of cycles. So it suffices to write any cycle as a product of 2-cycles:

$$(a_1\, a_2\, \ldots\, a_k) = (a_1\, a_2)(a_2\, a_3) \ldots (a_{k-1}\, a_k)$$

as is readily checked. ■

IV.2.12 Remark. Theorem IV.2.11 is equivalent to the following statement, which should be intuitively clear: A row of n objects can be placed in an arbitrary order by repeatedly interchanging pairs (i.e., performing transpositions). The proof even provides an upper bound for the minimal number of required transpositions: Supposed that a permutation σ can be written as a product of r disjoint ℓ_i-cycles, with $\ell_i \geq 1$ and $\sum \ell_i = n$. For an ℓ_i-cycle the proof of Theorem IV.2.11 shows that $\ell_i - 1$ interchanges suffice. In total we therefore obtain the upper bound $\sum(\ell_i - 1) = n - r$.

IV.2.13 Remark. It is even possible to show that any permutation can be written as a product of transpositions of a special kind. For example, we can achieve this using only 2-cycles of the form $(i\ i+1)$: If $i<j$, then

$$(i\ j)=(i\ i+1)(i+1\ i+2)\dots(j-1\ j)(j-2\ j-1)\dots(i\ i+1).$$

Alternatively, one can write any permutation as a product of 2-cycles $(1\ i)$. This follows from the observation that for $1\neq i\neq j\neq 1$ one has $(i\ j)=(1\ i)(1\ j)(1\ i)$. In other words, a row of objects can be put in arbitrary order by merely interchanging one given element consecutively with suitable other elements.

IV.3 Even and odd permutations

The presentation of a permutation as a product of 2-cycles is far from unique. For instance, if $\sigma=\tau_1\cdots\tau_r\in S_n$, where the τ_i are transpositions, then we can also write $\sigma=\tau_1\cdots\tau_r(1\,2)(1\,2)$. Nevertheless, we will show that the *parity* of the number of 2-cycles needed to represent a given permutation σ is completely determined by σ. It turns out that this parity, which we will call the sign of σ, is an important invariant of σ, which can be used to partition S_n into even and odd permutations.

We start by introducing some notation. This will be used to give a definition of the sign which is suitable for analyzing its properties.

IV.3.1 Notation. 1. For $n\geq 2$ write $X:=\{(i,j)\in\mathbb{Z}\times\mathbb{Z}\mid 1\leq i<j\leq n\}$.
2. For $\sigma\in S_n$ define $f_\sigma : X\to X$ by $f_\sigma(i,j)=(\min\{(\sigma(i),\sigma(j)\},\max\{(\sigma(i),\sigma(j)\})$.
3. Finally, define $h_\sigma : X\to\mathbb{Q}$ by $h_\sigma(i,j)=\dfrac{\sigma(j)-\sigma(i)}{j-i}$.

Some useful properties of these functions are as follows.

IV.3.2 Lemma. *Let $n\geq 2$. Then the following hold.*

1. *For $\sigma,\tau\in S_n$ one has $f_{\sigma\tau}=f_\sigma\circ f_\tau$.*
2. *The map f_σ is a bijection on X.*
3. *We have $\prod\limits_{(i,j)\in X} h_\sigma(i,j)=\pm 1$.*

Proof. 1. Both functions map an arbitrary pair $(i,j)\in X$ either to $(\sigma\tau(i),\sigma\tau(j))$ or to $(\sigma\tau(j),\sigma\tau(i))$ (depending on which of the two is in X). So the functions coincide.
2. This follows from $f_\sigma\circ f_{\sigma^{-1}}=f_{\sigma^{-1}}\circ f_\sigma=f_{\mathrm{id}}=\mathrm{id}$.
3. It suffices to show that the absolute value of the given product equals 1. This absolute value is equal to

$$\left(\prod_{(i,j)\in X}|\sigma(j)-\sigma(i)|\right)/\left(\prod_{(i,j)\in X}(j-i)\right).$$

Here the numerator is the product of all $(\ell-k)$, for $(k,\ell)=f_\sigma(i,j)\in f_\sigma(X)=X$. So numerator and denominator are equal. ∎

IV.3.3 Definition. We define the *sign* of a permutation $\sigma\in S_n$ by

$$\epsilon(\sigma):=\prod_{(i,j)\in X}h_\sigma(i,j)=\prod_{1\leq i<j\leq n}\frac{\sigma(j)-\sigma(i)}{j-i}=\pm 1$$

in case $n\geq 2$, and by $\epsilon(\sigma):=1$ for $n=1$. We call σ *even* if $\epsilon(\sigma)=1$ and *odd* if $\epsilon(\sigma)=-1$.

IV.3.4 Remark. We will soon describe an efficient way to calculate the sign of a permutation. Only using the definition, this can be quite elaborate; as an example, try to determine the sign of the 3-cycles (1 3 5) and (1 6 12).

The sign of a permutation may be interpreted as follows: the denominator of the expression defining the sign is a product of positive integers $j-i$. The factors of the numerator have the form $\sigma(j)-\sigma(i)$, and this factor is negative precisely when σ swaps the order of i and j, i.e. when $\sigma(j)<\sigma(i)$. If this occurs for an *even* number of pairs $(i,j)\in X$, then the sign $\epsilon(\sigma)=1$; if it happens for an *odd* number of pairs, then σ has sign -1.

The set $\{+1,-1\}$ is a group with respect to the usual multiplication. So ϵ is a map from the group S_n to the group $\{\pm 1\}$.

IV.3.5 Theorem. *The sign $\epsilon : S_n \longrightarrow \{\pm 1\}$ is a homomorphism.*

Proof. Let $\sigma,\tau\in S_n$. Then

$$\prod_{1\le i<j\le n}\frac{\sigma(\tau(j))-\sigma(\tau(i))}{\tau(j)-\tau(i)}=\prod_{(i,j)\in X}h_\sigma(f_\tau(i,j))=\prod_{(i,j)\in X}h_\sigma(i,j)=\epsilon(\sigma),$$

since f_σ is bijective on X by Lemma IV.3.2. Hence

$$\begin{aligned}\epsilon(\sigma\tau) &= \textstyle\prod_{(i,j)\in X}\frac{(\sigma\tau)(j)-(\sigma\tau)(i)}{j-i}\\ &= \Big(\textstyle\prod_{1\le i<j\le n}\frac{\sigma(\tau(j))-\sigma(\tau(i))}{\tau(j)-\tau(i)}\Big)\Big(\textstyle\prod_{(i,j)\in X}\frac{\tau(j)-\tau(i)}{j-i}\Big)\\ &= \epsilon(\sigma)\epsilon(\tau).\end{aligned}$$

■

In order to be able to use this result for computing the sign of permutations, we first prove a lemma, which will be used quite a few times throughout the course.

IV.3.6 Lemma.

1. We have $\rho(a_1\,a_2\,\ldots\,a_\ell)\rho^{-1}=(\rho(a_1)\,\rho(a_2)\,\ldots\,\rho(a_\ell))$ for any $\rho\in S_n$ and any ℓ-cycle $(a_1\,a_2\,\ldots\,a_\ell)\in S_n$.
2. Every transposition is odd.

Proof. 1. We have

$$(\rho(a_1\,\ldots\,a_\ell)\rho^{-1})(\rho(a_\ell))=(\rho(a_1\,\ldots\,a_\ell))(a_\ell)=\rho(a_1).$$

Similarly one finds $(\rho(a_1\,a_2\,\ldots\,a_\ell)\rho^{-1})(\rho(a_k))=\rho(a_{k+1})$ for $1\le k<\ell$. For all remaining $i\in\{1,\ldots,n\}$ one has $(\rho(a_1\,a_2\,\ldots\,a_\ell)\rho^{-1})(i)=i$. This shows the equality.
2. Let $(a_1\,a_2)$ be a transposition. Take any permutation ρ with $\rho(a_1)=1$ and $\rho(a_2)=2$. Then $\epsilon((1\,2))=\epsilon(\rho(a_1\,a_2)\rho^{-1})$. Since ϵ is a homomorphism, we have $\epsilon(\rho(a_1\,a_2)\rho^{-1})=\epsilon(\rho)\epsilon((a_1\,a_2))\epsilon(\rho)^{-1}=\epsilon((a_1\,a_2))$. So all 2-cycles have the same sign. For (1 2) we determine the sign from the definition: $\epsilon((1\,2))=-1$, because the only pair $(i,j)\in X$ changing order when (1 2) is applied, is (1,2). ■

IV.3.7 Corollary. *1. An ℓ-cycle σ has sign $\epsilon(\sigma)=(-1)^{\ell-1}$.*
2. If σ is a product of cycles of lengths $\ell_1,\ldots,\ell_r$, then $\epsilon(\sigma)=(-1)^{\sum_{i=1}^r(\ell_i-1)}$.
3. A permutation σ is even if and only if σ can be written as a product of an even number of 2-cycles.

Proof. 1. We have $(a_1\,a_2\,\ldots\,a_\ell)=(a_1\,a_2)(a_2\,a_3)\ldots(a_{\ell-1}\,a_\ell)$. The number of 2-cycles in this product is $\ell-1$, so because all of them have sign -1 and ϵ is a homomorphism, the result follows.
2. This is immediate from 1. since ϵ is a homomorphism.
3. Let $\sigma\in S_n$. By Theorem IV.2.11, σ can be written as a product of 2-cycles. The assertion now follows from 2. ■

IV.4 The alternating group

In this section we discuss even permutations in more detail. By definition, a permutations in S_n is even if and only if it lies in the kernel of the homomorphism $\epsilon\colon S_n \to \{\pm 1\}$, so it follows from Theorem IV.3.5 that the even permutations form a subgroup of S_n.

IV.4.1 Definition. For $n \geq 1$ the *alternating group* is the subgroup of S_n consisting of all even permutations. We denote it by A_n.

IV.4.2 Example. The group S_2 consists of the permutations (1) and (1 2), so we get $A_2 = \{(1)\}$.

The group S_3 consists of the identity, 2-cycles, and 3-cycles. The 2-cycles are not in A_3, and $A_3 = \{(1), (1\,2\,3), (1\,3\,2)\}$, which is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

In the group A_4 we find the identity, the 3-cycles (there are 8 of them) and the products of two disjoint 2-cycles (numbering 3). The group obtained in this way is not abelian (why?), and consists of 12 elements. ▬■

IV.4.3 Theorem. *For $n \geq 2$ the group A_n consists of $n!/2$ elements.*

Proof. The sets A_n and $(S_n \setminus A_n)$ are by definition disjoint, and their union is all of S_n. They have the same number of elements, because the map $\tau \mapsto (1\,2)\tau$ is a bijection between them. ■

IV.4.4 Theorem. For $n \geq 3$ the elements of A_n can be written as products of 3-cycles.

Proof. Let $\sigma \in A_n$. By Corollary IV.3.7 σ is a product of an even number of 2-cycles. In particular σ is a product of permutations $(a\ b)(c\ d)$. If $\{a,b\} = \{c,d\}$ the latter equals (1). If $\{a,b\}$ and $\{c,d\}$ have one element, say $a = c$ in common, then $(a\ b)(a\ d) = (a\ d\ b)$ is a 3-cycle. In the remaining case $(a\ b)(c\ d) = (a\ c\ b)(c\ d\ a)$. This shows the theorem. ■

IV.4.5 Example. We finish this chapter by illustrating Cayley's Theorem IV.1.4 for the example $G = (\mathbb{Z}/8\mathbb{Z})^*$. Since $\#G = \varphi(8) = 4$, the group G is isomorphic to a subgroup of S_4. We determine which subgroup the proof of Theorem IV.1.4 yields, and we even show that this subgroup is contained in A_4. The given proof identifies $a \in G$ with λ_a, the left-multiplication by a map. Moreover S_G is identified with S_4, simply by choosing a bijection between G and $\{1,2,3,4\}$. We choose the bijection $\overline{1} \mapsto 1, \overline{3} \mapsto 2, \overline{5} \mapsto 3$ en $\overline{7} \mapsto 4$.

The element $\overline{1} \in G$ gives rise to $\lambda_{\overline{1}} = \mathrm{id}_G$, which is the permutation (1). The element $\overline{3}$ yields the bijection $\lambda_{\overline{3}}$ on G sending $\overline{1}$ to $\overline{3}$, $\overline{3}$ to $\overline{3}\cdot\overline{3} = \overline{1}$, $\overline{5}$ to $\overline{3}\cdot\overline{5} = \overline{7}$, and $\overline{7}$ to $\overline{5}$. Using our bijection between G and $\{1,2,3,4\}$ this becomes the permutation $(1\,2)(3\,4)$.

A similar calculation sends $\overline{5}$ to the permutation $(1\,3)(2\,4)$ and $\overline{7}$ to $(1\,4)(2\,3)$. So apparently $(\mathbb{Z}/8\mathbb{Z})^*$ is isomorphic to the subgroup $\{(1), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ of S_4. This subgroup is contained in A_4 by Example IV.4.2. ▬■

IV.5 Exercises

1. Write each of the following permutations as a product of disjoint cycles:
   (a) $(3\,1\,4)(1\,5\,9\,2\,6)(5\,3)$
   (b) σ^{-1}, for $\sigma = (5\,6\,2)(1\,3)(1\,4)$.
2. (a) Find all $\sigma \in S_4$ satisfying $\sigma^2 = (1\,2)(3\,4)$.
   (b) Let $n > 1$. Does there exist $\sigma \in S_n$ such that $\sigma^2 = (1\,2)$?
   (c) Let $n \geq 6$. Does there exist $\sigma \in S_n$ such that $\sigma^2 = (1\,2)(3\,4\,5\,6)$?
3. Suppose σ is a k-cycle. Show that σ^n is a k-cycle if and only if $\gcd(k,n) = 1$.
4. (a) Determine which integers occur as the order of an element of S_6.
   (b) For each of the integers above, how many elements in S_6 have this order?
5. What is the least n such that $30 | \#S_n$? What is the least n such that S_n contains an element of order 30?
6. Determine the order and the sign of $(5\,6\,7\,8\,9)(3\,4\,5\,6)(2\,3\,4)(1\,2)$ in S_9.
7. (a) With $\sigma = (1\,2)(3\,4\,5)(6\,7\,8\,9\,10)$, write σ^{2016} as a product of disjoint cycles.
   (b) Do the same with τ^{2017}, given $\tau = (1\,2\,3)(3\,4)(4\,5\,6\,7)$.
8. Let $\sigma \in S_n$. Show that if $\sigma(1\,2\,\ldots\,n) = (1\,2\,\ldots\,n)\sigma$, then $\sigma = (1\,2\,\ldots\,n)^i$ some i.
9. For $n \geq 1$, determine the center $\mathcal{Z}(S_n)$ of S_n (see Exercise III.7).
10. (a) Let $\sigma, \tau \in S_n$. Show that if σ is a product of disjoint cycles of lengths $\ell_1, \ldots, \ell_r$, then so is $\tau\sigma\tau^{-1}$.
    (b) Vice versa, if $\sigma_1, \sigma_2 \in S_n$ are both products of disjoint cycles of lengths $\ell_1, \ldots, \ell_r$, show that $\tau \in S_n$ exists with $\sigma_2 = \tau\sigma_1\tau^{-1}$.
11. (a) Suppose $a \neq 1 \neq b$. Compute $(1\,a)(1\,b)(1\,a)(1\,b)$.
    (b) Show that every element of A_n can be written as a product of elements of the form $\sigma\tau\sigma^{-1}\tau^{-1}$, for $\sigma, \tau \in S_n$.
    (c) Show that if G is an abelian group and $f : S_n \to G$ a homomorphism, then $A_n \subset \ker(f)$.
    (d) Show that if $g : S_n \to S_m$ is a homomorphism, then $g(A_n) \subset A_m$.
12. A subgroup $H \subset S_n$ is called *transitive* if for every $\{i,j\} \subset \{1,2,\ldots,n\}$ some $\tau \in H$ exists with $\tau(i) = j$.
    (a) Show that for $n \geq 3$ the group A_n is a transitive subgroup of S_n.
    (b) Show that if G is a group and $\#G = n$, then the subgroup of S_n constructed in the proof of Cayley's theorem is a transitive subgroup of S_n.
    (c) Using Cayley's theorem, construct a transitive subgroup of S_6 which is isomorphic to S_3.

V GROUPS OF SYMMETRIES

In this chapter we consider groups consisting of special bijections on some space or set. This leads to the kind of groups which are of special interest in physics, or in some cases also in discrete mathematics. We are particularly interested in symmetries of geometric objects, such as polygons in the plane. The concepts from linear algebra which are used in this chapter may be found in essentially any textbook on the subject.

V.1 Some groups of matrices

The vector space $\mathbb{R}^2$ over $\mathbb{R}$ can be visualized as a plane. Using the theorem of Pythagoras, the standard interpretation of $\mathbb{R}^2$ as a plane allows us to introduce a distance function d on $\mathbb{R}^2$:

$$d((a,b),(c,d)) := \sqrt{(a-c)^2+(b-d)^2}.$$

One can define an analogous distance for $\mathbb{R}^3$, and in linear algebra this was generalized to the situation of an arbitrary (real or complex hermitian) inner product space $(V,\langle\cdot\,,\cdot\rangle)$. In the latter case the distance $d(v,w)$ between two vectors $v,w\in V$ is defined as

$$d(v,w) = ||v-w|| = \sqrt{\langle v-w, v-w\rangle}.$$

We can attach a group to such an inner product space.

V.1.1 Definition. Let $(V,\langle\cdot\,,\cdot\rangle)$ be a real or complex hermitian inner product space. The set of all linear maps $\varphi : V\to V$ satisfying $\langle v,w\rangle = \langle\varphi(v),\varphi(w)\rangle$ for all $v,w\in V$, is denoted by $O(V,\langle\cdot\,,\cdot\rangle)$.

V.1.2 Theorem. *Let $(V,\langle\cdot\,,\cdot\rangle)$ be as above and suppose that V is finite dimensional. Then the $(O(V,\langle\cdot\,,\cdot\rangle),\circ,\mathrm{id}_V)$ a group, where $\circ$ is composition of linear maps and id_V is the identity map on V.*

Proof. We first show that $O(V,\langle\cdot\,,\cdot\rangle)$ is a subset of the group $\mathrm{GL}(V)$ consisting of *all* invertible linear maps from V to itself. In other words, we show that the elements of $O(V,\langle\cdot\,,\cdot\rangle)$ are invertible. Let $\varphi\in O(V,\langle\cdot\,,\cdot\rangle)$. If $\varphi(v)=0$, then $\langle v,v\rangle = \langle\varphi(v),\varphi(v)\rangle = 0$, hence $v=0$. This implies that φ is injective. Since injective linear maps from a finite dimensional vector space to itself are automatically surjective, φ is invertible.

Hence, to show that $O(V,\langle\cdot\,,\cdot\rangle)$ is a group, it suffices to verify that it is a subgroup of $\mathrm{GL}(V)$. We leave this as an exercise to the reader. ∎

V.1.3 Remark. Note that the condition "V is finite dimensional" was used in the argument above to show that elements of $O(V,\langle\,\cdot\,,\,\cdot\,\rangle)$ are indeed invertible. In fact, for inner product spaces of infinite dimension over $\mathbb{R}$ or $\mathbb{C}$ this may not hold, as the following example shows. Take V to be the vector space over $\mathbb{R}$ consisting of all sequences $(a_n)_{n\geq 1}$ of real numbers, with the property that $a_n = 0$ for all but finitely many positive integers n. On V we define the 'standard' inner product $\langle (a_n)_{n\geq 1},(b_n)_{n\geq 1}\rangle = \sum_{n=1}^{\infty} a_n b_n$. (The sum is well defined since only finitely many terms are nonzero.) The shift operator $\sigma : V \to V$ given by

$$\sigma(a_1,a_2,a_3,\ldots) = (0,a_1,a_2,a_3,\ldots)$$

is an element of $O(V,\langle\,\cdot\,,\,\cdot\,\rangle)$, but it is clearly not invertible. So in this example $O(V,\langle\,\cdot\,,\,\cdot\,\rangle)$ is *not* a group.

If $A = (a_{i,j})$ is a (square) matrix with real or complex coefficients, then linear algebra defines the *adjoint* of A, notation A^*, as $A^* = (b_{i,j})$ with $b_{i,j} = \overline{a_{j,i}}$. So to obtain A^* one reflects all entries of the matrix with respect to the main diagonal, and then one takes the complex conjugate of the entries. A more intrinsic definition of A^* is that it is the unique matrix such that for all vectors v,w one has $\langle Av,w\rangle = \langle v,A^*w\rangle$ with respect to the standard inner product. More generally, if $\varphi \in \mathrm{GL}(V)$ is given by a matrix A with respect to an orthonormal basis of V, then $\varphi \in O(V,\langle\,\cdot\,,\,\cdot\,\rangle)$ if and only if $A^*A = I_n$, where I_n is the unit matrix. This translates the group $O(V,\langle\,\cdot\,,\,\cdot\,\rangle)$ into a group of matrices, namely into a subgroup of $\mathrm{GL}_n(\mathbb{R})$ of $\mathrm{GL}_n(\mathbb{C})$ with $n = \dim(V)$. We now duscuss some groups of matrices and some relevant subgroups obtained in this way.

V.1.4 Definition. Let $n \in \mathbb{Z}, n > 0$. We define

1. the *orthogonal* group $\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^*A = I\}$;
2. the *unitary* group $\mathrm{U}(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid A^*A = I\}$;
3. the *special orthogonal* group $\mathrm{SO}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^*A = I \text{ and } \det(A) = 1\}$;
4. the *special unitary* group $\mathrm{SU}(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid A^*A = I \text{ and } \det(A) = 1\}$.

V.1.5 Example. For $n = 1$ we obtain $\mathrm{O}(1) = \{a \in \mathbb{R}\setminus\{0\} \mid a^2 = 1\} = \{\pm 1\}$. As maps on $\mathbb{R}$ these are the identity and 'taking the opposite'. The groups SO(1) and SU(1) both equal the trivial group consisting of only one element. The group U(1) is more interesting: it is the group of all points on the unit circle in $\mathbb{C}$, with multiplication as group law, see Exercise III.10.

The group SO(2) consists of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a,b,c,d \in \mathbb{R}$ satisfying $a^2 + c^2 = b^2 + d^2 = ad - bc = 1$ and $ab + cd = 0$. Writing $a = \cos\alpha$ and $c = \sin\alpha$, it follows that $d = \cos\alpha$ and $b = -\sin\alpha$. So as a map from $\mathbb{R}^2$ to itself this matrix represents the counterclockwise rotation with center $(0,0)$ by an angle α. The group SO(2) is exactly the group consisting of all such rotations.

In the group O(2) we also find the matrices of the form $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha \end{pmatrix}$. Geometrically this represents a reflection in the line passing through the origin, which intersects the positive x-axis with an angle $-\alpha/2$. So O(2) consists of geometrically defined maps, namely all rotations with the origin as center, and all refections in lines passing through the origin. Note that O(2) is not commutative: if we first reflect in the x-axis and then rotate (counter-clockwise) by 90 degrees, $(0,1)$ is the image of $(1,0)$. However applying the maps in the reverse order maps $(1,0)$ to $(0,-1)$. ∎

All groups given in Definition V.1.4 may be regarded as groups of invertible linear maps from $\mathbb{R}^n$ or $\mathbb{C}^n$ to itself, with the property that they preserve the standard inner product and therefore also the distance between points. From now on

we will restrict ourselves to the real case, with a focus on $\mathbb{R}^2$ and $\mathbb{R}^3$. Geometrically, the fact that a map is distance preserving means, that for example a triangle is mapped to a congruent triangle, because the three vertices are mapped to three new points which pairwise have the same distance as the original ones. A point on one of the sides is mapped to a point which has the same distances to the new vertices as the original point had to the old ones. This implies that the sides of the original triangle are mapped to sides of the new one. This argument shows that all distance preserving maps (we do not need to assume linearity) map lines to lines and angles to equally large angles.

V.2 Groups of isometries

We work with the space $\mathbb{R}^n$, equipped with the norm $||(a_1,\ldots,a_n)|| := \sqrt{a_1^2+\ldots+a_n^2}$ and the distance $d(v,w) := ||v-w||$ for $v,w \in \mathbb{R}^n$.

V.2.1 Definition. An *isometry* on $\mathbb{R}^n$ is a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with the property $d(v,w) = d(\varphi(v),\varphi(w))$ for all $v,w \in \mathbb{R}^n$.

V.2.2 Example. Translations, rotations, and reflections in a point or in a line or in a plane are examples of isometries. ▬

V.2.3 Theorem.

1. *An isometry on $\mathbb{R}^n$ mapping $0 \in \mathbb{R}^n$ to 0 is linear.*
2. *The linear isometries on $\mathbb{R}^n$ are exactly the elements of $O(n)$.*
3. *Every isometry can be written as a composition of a translation and a linear isometry.*
4. *Isometries are invertible.*

Proof. 1. We have $||u-v||^2 = \langle u-v,u-v\rangle = ||u||^2+||v||^2-2\langle u,v\rangle$ for $u,v \in \mathbb{R}^n$. If φ is an isometry and $\varphi(0)=0$, then

$$\begin{aligned} 2\langle u,v\rangle &= ||u-0||^2+||v-0||^2-||u-v||^2 \\ &= ||\varphi(u)-\varphi(0)||^2+||\varphi(v)-\varphi(0)||^2-||\varphi(u)-\varphi(v)||^2 \\ &= 2\langle\varphi(u),\varphi(v)\rangle. \end{aligned}$$

A calculation now shows $||\varphi(u+av)-\varphi(u)-a\varphi(v)||^2 = 0$ for $a \in \mathbb{R}$, so φ is linear.
2. The proof of 1. shows that a linear isometry preserves the inner product, so is in $\mathrm{O}(n)$. Vice versa, an element $A \in \mathrm{O}(n)$ is clearly linear. It is an isometry since

$$||v-w||^2 = \langle v-w,v-w\rangle = \langle A(v-w),A(v-w)\rangle = ||A(v-w)||^2 = ||A(v)-A(w)||^2$$

3. Let φ be an isometry. Write $v = \varphi(0)$. Define $\tau_v : \mathbb{R}^n \to \mathbb{R}^n$ to be translation by v, so $\tau_v(w) = v+w$ for $w \in \mathbb{R}^n$. Moreover define $\psi : \mathbb{R}^n \to \mathbb{R}^n$ by $\psi(w) = \varphi(w)-v$. Then τ_v and ψ are isometries, and $\psi(0)=0$, so from 1. we see that ψ is linear. One has

$$\varphi(w) = \varphi(w)-v+v = \psi(w)+v = \tau_v(\psi(w))$$

for all $w \in \mathbb{R}^n$, so $\varphi = \tau_v \circ \psi$.
4. By 3. and the fact that composing bijections yields a bijection, it suffices to show that translations and linear isometries are invertible. This is clear for translations, and the proof of Theorem V.1.2 shows it for linear isometries. ■

Because of this theorem linear isometries are the same as the *orthogonal* (linear) maps studied in linear algebra. For $\mathbb{R}^2$ we determined these maps in Example V.1.5: all reflections in a line through the origin, and the rotations with center the origin.

We now consider isometries which, when restricted to a subset $\mathbb{R}^n$, map this subset to itself. It is not hard to show that, for a given subset of $\mathbb{R}^n$, the set of such isometries forms a subgroup of the group of *all* isometries.

V.2.4 Definition. The *symmetry group* of a subset $F \subset \mathbb{R}^n$ is defined as the group of all all isometries on $\mathbb{R}^n$ with the property that F is mapped to F.

It turns out that up to isomorphism the symmetry group of a set F is not affected by the *position* of F in $\mathbb{R}^n$, only by 'the shape of F':

V.2.5 Theorem. *If $F \subset \mathbb{R}^n$, $a \in \mathbb{R}_{>0}$, and φ is an isometry on $\mathbb{R}^n$, then the symmetry group of $a\varphi(F)$ and of F are isomorphic.*

Proof. The map $\sigma \mapsto a\varphi\sigma\varphi^{-1}\frac{1}{a}$ sends the symmetry group of F to that of $a\varphi(F)$ (compare Exercise 3), and this map is a homomorphism. It is bijective with inverse given by $\tau \mapsto \varphi^{-1}\frac{1}{a}\tau a\varphi$. ∎

We now describe the symmetry groups of certain subsets of $\mathbb{R}^2$. A major role will be played by the following result.

V.2.6 Lemma. *If G is a subgroup of $SO(2)$ consisting of exactly N elements, then G consists of all rotations by multiples of $2\pi/N$. In particular $G \cong \mathbb{Z}/N\mathbb{Z}$.*

Proof. Every element of SO(2), so in particular every element of G, is a rotation. Let $\sigma \in \mathrm{SO}(2)$ be the rotation by the smallest possible positive angle $2\pi\alpha$ such that σ is in G. Since G is finite, $\sigma^n = \mathrm{id}$ for some $n > 0$, so $n \cdot 2\pi\alpha$ is an integral multiple of 2π. This implies $\alpha \in \mathbb{Q}$, so we may write $\alpha = a/b$ for positive integers a, b with $\gcd(a,b) = 1$. Take $c, d \in \mathbb{Z}$ with $ac + bd = 1$, then σ^c is the rotation by $2\pi ac/b = 2\pi(1 - bd)/b$, i.e., by an angle $2\pi/b$. Since $2\pi a/b$ is the least positive angle of rotation in G, we have $a = 1$. We now show $b = N$. Take an arbitrary rotation $\tau' \in G$ by an angle $2\pi\ell/m$. By the same reasoning used above, we find a power τ of τ' representing rotation by $2\pi/m$. Note that τ' is a power of σ if and only if τ is some power of σ. By taking a suitable combination $\sigma^p\tau^q$ we find an element of G representing rotation by $2\pi/\mathrm{lcm}(b,\ell)$. The minimality of $2\pi/b$ shows $\mathrm{lcm}(b,\ell) \leq b$, so $\ell | b$. This implies that every element of G is a power of σ. So $N = \#G = \mathrm{ord}(\sigma) = b$, proving the lemma. ∎

An alternative, more geometric proof runs as follows. Take a circle around the origin and a point on it. The images of this point under the elements of G yield N points on the circle. Using that G consists of isometries, one can show that these N points are the vertices of a regular N-gon. The rotations permuting these vertices now form the group G.

V.3 The dihedral groups.

Let $C_r \subset \mathbb{R}^2$ be the circle with radius r around the origin. An isometry mapping C_r to itself necessarily fixes the center: namely, any point of C_r has a unique point of C_r at distance $2r$ (the antipodal point). As a result, symmetries of C_r will map lines through the origin to lines through the origin, and therefore the intersection point of these lines will be fixed. We conclude that the symmetry group of the circle is isomorphic to the group O(2).

V.3.1 Definition. The symmetry group of the circle C_r is called the *infinite dihedral group*. This group is denoted by D_∞.

V.3.2 Theorem. *The group D_∞ is isomorphic to O(2), and consists of reflections σ across arbitrary lines through the center of the circle, and of all rotations ρ around the center of the circle. The subset $R \subset D_\infty$ of all rotations is a commutative subgroup of D_∞.*

If $\sigma \in D_\infty$ is any reflection, then

$$D_\infty = R \cup R \cdot \sigma.$$

Taking σ the reflection across the x-axis, we have $\sigma\rho\sigma = \rho^{-1}$ for any $\rho \in R$.

Proof. We already saw that $D_\infty \cong \mathrm{O}(2)$ and that O(2) consists of reflections and rotations. The rotations are the matrices in O(2) of determinant 1, so R is the kernel of the homomorphism "determinant": $\mathrm{O}(2) \to \{\pm 1\}$.

The elements in O(2) having determinant -1 are reflections (their characteristic polynomial has the form $X^2 - tX - 1$ for some $t \in \mathbb{R}$, so we have two real eigenvalues. They have absolute value 1 (since the matrix is orthogonal) and product -1. Hence the matrix represents the reflection across the line spanned by the eigenvector with eigenvalue $+1$.

The partition $D_\infty = R \cup R \cdot \sigma$ is the partition of D_∞ into rotations (determinant 1) and reflections (determinant -1).

The reflection σ in the x-axis is given by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. For an arbitrary rotation $\rho = \left(\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\right)$ one computes

$$\sigma\rho\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \rho^{-1}$$

since $\cos(-\alpha) = \cos(\alpha)$ and $\sin(-\alpha) = -\sin(\alpha)$. This proves the theorem. ∎

When computing in D_∞ it is often convenient to regard rotations and reflections as maps on the complex plane $\mathbb{C}$. "Reflection in the x-axis" then becomes complex conjugation

$$c : z \mapsto \overline{z}$$

and "rotation by α" becomes multiplication by $e^{\alpha i}$, i.e.

$$m : z \mapsto e^{\alpha i} \cdot z.$$

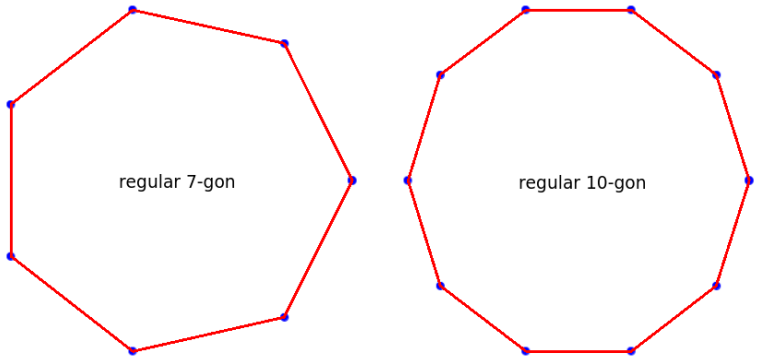
As an example, cmc is the map sending z to $cmc(z) = cm(\overline{z}) = c(e^{\alpha i}\overline{z}) = \overline{e^{\alpha i}\overline{z}} = e^{-\alpha i}z$, so $cmc = m^{-1}$ as we saw earlier.

Subdividing the circle into $n \geq 2$ equal segments yields n vertices which define a regular n-gon F_n.

V.3.3 Definition. The symmetry group of F_n is called the *n-th dihedral group D_n*.

The nth dihedral groups will serve as standard examples of finite groups in the rest of this course, alongside the groups $\mathbb{Z}/N\mathbb{Z}$ and $(\mathbb{Z}/N\mathbb{Z})^\times$, and the permutation groups and their subgroups. Their geometric interpretation often makes it possible to visualize abstract results in a concrete way.

The center of F_n is fixed under all symmetries, so D_n is a subgroup of $D_\infty = \mathrm{O}(2)$. The group D_n consists of rotations and reflections; the rotations in D_n are those by an angle $k \cdot 2\pi/n$, for $0 \leq k < n$. There are n of these. The rotation by the least positive angle $2\pi/n$ we denote by ρ. The reflections in D_n are precisely the reflections in either lines containing the origin and a vertex of F_n, or lines containing the



origin and the midpoint of an edge of F_n. One of these reflections is the reflection in the x-axis, which from now on we denote by σ. There are precisely n reflections in D_n, namely all $\sigma\rho^k$ for $0 \le k < n$. So D_n is a finite group consisting of $n + n = 2n$ elements.

We summarize this discussion as follows.

V.3.4 Theorem. *The group D_n consists of $2n$ elements. It is abelian if and only if $n = 2$.*

The group D_n contains the rotation ρ by an angle $2\pi/n$ and the reflection σ in the x-axis. Every element of D_n can be written in a unique way as ρ^k or $\sigma\rho^k$, for some $0 \le k < n$.

One has $\mathrm{ord}(\rho) = n$ *and* $\mathrm{ord}(\sigma\rho^k) = 2$*, so in particular $\rho^n = \sigma^2 = id$. Moreover, $\sigma\rho\sigma = \rho^{-1}$.*

The subgroup R_n of D_n consisting of all rotations is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Proof. The inverse ρ^{-1} is a rotation by an angle $(n-1)2\pi/n$. For $n > 2$ this differs from a rotation by $2\pi/n$, so then $\sigma\rho\sigma = \rho^{-1} \neq \rho$ by Theorem V.3.2. This implies that D_n is not commutative if $n > 2$.

The remaining assertions in the theorem follow immediately from the definitions, the above discussion and Theorem V.3.2. ∎

V.3.5 Example. For $n = 2$ the group D_2 consists of 4 elements. In this case ρ is the map "rotate by 180 degrees", so $\rho(x, y) = (-x, -y)$. In particular $\rho^{-1} = \rho$, hence $\sigma\rho = \rho\sigma$. Therefore, D_2 is commutative. We already knew this, because all groups consisting of exactly 4 elements are commutative by Exercise III.15. In the present case $\sigma\rho$ is the reflection across the y-axis. All nontrivial elements of D_2 have order 2 which implies $D_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Note that this example is somewhat odd: a regular 2-gon is just a line segment. The reflection across the line containing this segment is an element of order 2 in D_2. However, this reflection fixes every point in the segment F_2. ▬

V.3.6 Example. Let us determine the integers $n > 0$ for which the rotation r by 180 degrees given by $r(x, y) = (-x, -y)$, is an element of D_n.

We know that $\mathrm{ord}(r) = 2$. If $r \in D_n$ is a rotation, then $r = \rho^k$ for some k. Therefore, $r^n = \rho^{nk} = id$, so $2 = \mathrm{ord}(r)$ is a divisor of n and thus n is even. Vice versa, let $n = 2m$ for some integer $m > 0$. Then ρ^m is a rotation with $\mathrm{ord}(\rho^m) = 2$.Thus, ρ^m is a rotation by 180 degrees: $\rho^m = r$. We conclude that

$$r \in D_n \Leftrightarrow n \text{ is even.}$$

Note that r is in the *center* of D_{2m}, i.e. we have $r\tau = \tau r$ for all $\tau \in D_{2m}$. ▬

V.4 Symmetries of a strip: frieze groups

This section discusses the symmetries of a strip in the plane, i.e., of the set of points in $\mathbb{R}^2$ located between two parallel lines. In particular so-called *discrete* subgroups of this symmetry group will be presented. It turns out that this topic is related to art and to architecture.

Using Theorem V.2.5 the width of the strip can be scaled without changing the symmetry group in an essential way. Moreover we may change the strip by applying an isometry. This observation shows that the following choice of strip and group describes in some sense all cases.

V.4.1 Definition. By G_S we denote the group of symmetries of the set $S \subset \mathbb{R}^2$ defined by

$$S := \{(x,y) \in \mathbb{R}^2 \mid -1 \le y \le 1\}.$$

We start by presenting a more explicit description of the group G_S.

V.4.2 Theorem. The group G_S of all symmetries of the strip S consists of all isometries $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\varphi(x,y) = (\pm x + u, \pm y)$, for all four possibilities of the signs $(\pm,\pm)$ and all $u \in \mathbb{R}$.

Proof. Using Theorem V.2.3 and Example V.1.5 we find that the isometries φ of $\mathbb{R}^2$ are given by $\varphi(x,y) = (ax+by+u, cx+dy+v)$ with $a,b,c,d,u,v \in \mathbb{R}$ satisfying $a^2+c^2 = 1 = b^2+d^2$ and $ab+cd = 0$. If φ sends S to S, then in particular it sends the x-axis to itself. This means that the second coordinate of $\varphi(x,0) = (ax+u, cx+v)$ equals 0 for every $x \in \mathbb{R}$. From this one concludes $c = v = 0$. Hence $a^2 = 1 = b^2 + d^2$ and $ab = 0$, which implies $b = 0$ and $a^2 = 1 = d^2$. So indeed φ has the required form, and it is easy to verify that all isometries of this form send S to S, so they are in G_S. ∎

Here are the 4 types of elements in the symmetry group of the strip S:

- $\tau_u : (x,y) \mapsto (x+u, y)$ is the *translation* by $(u,0)$. We have $\mathrm{ord}(\tau_u) = \infty$ except when $u = 0$; obviously τ_0 is the identity map, which has order 1.
- $\rho_u : (x,y) \mapsto (-x+u, y)$ is the *reflection* across the vertical line given by $x = \frac{1}{2}u$. Since ρ_u^2 is the identity map and ρ_u is not, one finds $\mathrm{ord}(\rho_u) = 2$. Put $\rho := \rho_0$, then $\tau_u \rho = \rho_u = \rho \tau_{-u}$. In particular, all of these reflections can be expressed as a product of ρ and a translation.
- $\gamma_u : (x,y) \mapsto (x+u, -y)$ is called a *glide reflection*; in the case $u = 0$ it is the reflection $\gamma = \gamma_0$ across the x-axis. In general it is the composition of this reflection and a translation by $(u,0)$: we have $\gamma\tau_u = \gamma_u = \tau_u\gamma$. Since $\gamma_u^2 = \tau_{2u}$ one finds $\mathrm{ord}(\gamma_0) = 2$ and $\mathrm{ord}(\gamma_u) = \infty$ whenever $u \neq 0$.
- $\pi_u : (x,y) \mapsto (-x+u, -y)$ is the *point reflection* with center $(\frac{1}{2}u, 0)$. Clearly $\mathrm{ord}(\pi_u) = 2$. In terms of the point reflection $\pi = \pi_0$ in the origin, we have $\pi\tau_{-u} = \pi_u = \tau_u\pi$.

In particular, the description above shows that every element of G_S can be written as a product of a translation and an element in $\{\mathrm{id}, \rho, \gamma, \pi\}$. Note that $\{\mathrm{id}, \rho, \gamma, \pi\}$ is in fact a commutative subgroup of G_S: each of these four elements is its own inverse, and $\rho\gamma = \pi = \gamma\rho$ and $\rho\pi = \gamma = \pi\rho$ and $\gamma\pi = \rho = \pi\gamma$. So in fact this subgroup is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We summarize and extend this discussion in the following result.

V.4.3 Lemma. *The sets $T = \{\tau_u \mid u \in \mathbb{R}\}$ and $H = \{\mathrm{id}, \rho, \gamma, \pi\}$ are subgroups of G_S. Every element $g \in G_S$ can be expressed in a unique way as $g = th$ for some $t \in T$*

and some $h \in H$. Similarly, every $g \in G_S$ can be expressed in a unique way as $g = h't'$ for some $t' \in T$ and some $h' \in H$. In other words, $G_S = TH = HT$. The map $\varphi : G_S \longrightarrow H$ defined by $g = th \mapsto \varphi(g) := h$ is a surjective group homomorphism with kernel T.

Proof. We already observed that $H \subset G_S$ is a subgroup. The fact that also $T \subset G_S$ is a subgroup is easy to verify. The discussion preceding this lemma shows that every element of G_S can be written as th and as $h't'$ for some $t, t' \in T$ and some $h, h' \in H$. Since $g(0,0) = th(0,0) = t(0,0)$ it follows that t is the translation by $g(0,0)$, hence indeed g uniquely determines t and therefore also $h = t^{-1}g$. Similarly, if $g = h't'$ then $g^{-1} = t'^{-1}h'$ which determines t'^{-1} and hence t', as well as h'.

To see that φ is a homomorphism, first observe, as shown before, that for every $h \in H$ and every translation $t \in T$ there is another translation $\tilde{t} \in T$ (which is either t itself or its inverse) such that $ht = \tilde{t}h$. Hence, with $g = th$ and $g' = t'h'$ one finds $g'g = t'(h't)h = t'\tilde{t}h'h$. As a consequence, $\varphi(g'g) = h'h = \varphi(g')\varphi(g)$, which shows that φ is a homomorphism. It is clearly surjective, and has kernel T. ∎

Using the notation for elements of G_S as introduced above, the next equalities in G_S are not hard to verify.

(a) $\tau_u\tau_v = \tau_{u+v}$ and in particular $\tau_u^{-1} = \tau_{-u}$.
(b) $\rho_u\rho_v = \tau_{u-v}$ and $\rho_u^{-1} = \rho_u$ and $\rho_u\tau_v = \rho_{u-v} = \tau_{-v}\rho_u$.
(c) $\gamma_u\gamma_v = \tau_{u+v}$ and $\gamma_u^{-1} = \gamma_{-u}$.
(d) $\gamma_u\tau_v = \gamma_{u+v} = \tau_v\gamma_u$ and $\gamma_u\rho_v = \pi_{u+v} = \rho_v\gamma_{-u}$.
(e) $\pi_u\pi_v = \tau_{u-v}$ and $\pi_u^{-1} = \pi_u$.
(f) $\pi_u\tau_v = \pi_{u-v} = \tau_{-v}\pi_u$ and $\pi_u\rho_v = \gamma_{u-v} = \rho_{-v}\pi_{-u}$ and $\pi_u\gamma_v = \rho_{u-v} = \gamma_{-v}\pi_u$.

V.4.4 Definition. A *frieze group* is a subgroup $F \subset G_S$ with the property $F \cap T \cong \mathbb{Z}$.

Note that if $F \subset G_S$ is a frieze group, then by fixing an isomorphism $f : \mathbb{Z} \to F \cap T$ we have $t := f(1)$ is a translation in F, so $t = \tau_u$ for some $u \in \mathbb{R}$. Using that f is a homomorphism we have $f(n) = t^n = \tau_{nu}$. By assumption *every* translation in F is obtained in this way, in other words: $F \cap T = \{\tau_{nu} \mid n \in \mathbb{Z}\}$ and $|u| > 0$ is the minimal positive number such that the translation $\tau_{|u|}$ by $(|u|, 0)$ is in the frieze group F.

V.4.5 Example. $F := \{\tau_n \mid n \in \mathbb{Z}\}$ is a frieze group. By definition it consists of the translations by all points $(n, 0)$ for $n \in \mathbb{Z}$.

Also $F' := \{\gamma_1^n \mid n \in \mathbb{Z}\}$ is a frieze group. The glide reflections $\gamma_{2m+1} = \gamma_1^{2m+1}$ for $m \in \mathbb{Z}$ are in F', as well as all translations $\tau_{2m} = \gamma_1^{2m}$ for $m \in \mathbb{Z}$. In this case an isomorphism $\mathbb{Z} \cong F' \cap T$ is provided by $m \mapsto \gamma_1^{2m}$. ∎

We now present an alternative definition of "frieze groups". To achieve this, we first define a particular type of subgroup of the group $\mathrm{Isom}(\mathbb{R}^n)$ of all isometries of $\mathbb{R}^n$.

V.4.6 Definition. A subgroup $G \subset \mathrm{Isom}(\mathbb{R}^n)$ is called *discrete* if for every $v \in \mathbb{R}^n$ the ball $B_v := \{w \in \mathbb{R}^n \mid d(v, w) \leq 1\}$ has the following property:

$$\{g \in G \mid g(B_v) \cap B_v \neq \varnothing\}$$

is finite.

To understand this definition, observe that B_v is the n-dimensional ball with radius 1 and center $v \in \mathbb{R}^n$. Its image $g(B_v)$ under any isometry of $\mathbb{R}^n$ equals the ball $B_{g(v)}$. These balls have an empty intersection precisely when $||v - g(v)|| > 2$. So G is discrete, precisely when every $v \in \mathbb{R}^n$ has the property that only finitely many $g \in G$ send v to a point $g(v)$ at distance less than or equal to 2 from v.

V.4.7 Example. Clearly every finite subgroup of $\mathrm{Isom}(\mathbb{R}^n)$ is discrete. The infinite dihedral group D_∞ discussed in Definition V.3.1 and Theorem V.3.2 is *not* discrete: the group is infinite, and every element of it fixes the origin.

The translations $\tau_n \in G_S$ by a point $(n,0)$ with $n \in \mathbb{Z}$ define an infinite discrete subgroup of $\mathrm{Isom}(\mathbb{R}^2)$: namely, for any $v \in \mathbb{R}^2$ and any $n \in \mathbb{Z}$ we have $||v-\tau_n(v)|| = |n|$, so only $\tau_0, \tau_{\pm 1}$, and $\tau_{\pm 2}$ send v to a point at distance ≤ 2 from v. ▬■

V.4.8 Theorem. *A subgroup $F \subset G_S$ is a frieze group if and only if F is infinite and discrete.*

Proof. $\Rightarrow$: Assume $F \subset G_S$ is a frieze group. By definition $F \cap T \cong \mathbb{Z}$, so $F \cap T$ is infinite and therefore F is infinite, too. To verify that F is discrete we will use the homomorphism $\varphi : G_S \to H = \{\mathrm{id}, \rho, \gamma, \pi\}$. Write $n := \#\varphi(F)$ and $K := F \cap T$. Let $f_1 = \mathrm{id}, \ldots, f_n$ be elements of F such that $\varphi(F) = \{\varphi(f_1) = \mathrm{id}, \ldots, \varphi(f_n)\}$. Then $F = Kf_1 \cup \ldots \cup Kf_n$. As we observed earlier, F being a frieze group implies that K consists of all translations τ_{mc} for some fixed $c > 0$ with m ranging over the integers. Now let $v \in \mathbb{R}^2$ and take $f \in F$. Write $f = \tau_{mc} f_i$ for some $m \in \mathbb{Z}, i \in \{1, \ldots, n\}$. Then $f(B_v) = B_{f_i(v)+m(c,0)}$ so clearly $f(B_v) \cap B_v \neq \varnothing$ is only possible for finitely many values of m. Therefore F is discrete.
$\Leftarrow$: Now we assume that $F \subset G_S$ is an infinite discrete subgroup. The argument above shows that $F = Kf_1 \cup \ldots \cup Kf_n$ with $K = F \cap T$ and $f_1, \ldots, f_n \in F$. Since F is infinite, at least one (and therefore, all) of the sets Kf_i are infinite, so $\#K = \infty$. We have that F is discrete, and therefore its subgroup K is discrete as well. So we have a discrete group K consisting of translations by points $(c,0)$, and what remains to be shown is that $K \cong \mathbb{Z}$. The definition of discreteness applied to K and to the ball $B_{(a,0)}$ of radius 1 and center $(a,0)$ shows, that K contains only finitely many translations τ_c such that $|c-a| \leq 2$. In other words, every (closed) interval of length 4 in $\mathbb{R}$ contains only finitely many $c \in \mathbb{R}$ such that $\tau_c \in K$. As K is not empty, this implies that we can take the smallest possible $c > 0$ with $\tau_c \in K$. Claim: $K = \{\tau_{mc} | m \in \mathbb{Z}\}$ and $m \mapsto \tau_{mc}$ is an isomorphism $\mathbb{Z} \cong K$. Namely, by definition $\tau_c \in K$ hence since K is a group, for all $m \in \mathbb{Z}$ also $\tau_{mc} = \tau_c^m \in K$. This shows $K \supset \{\tau_{mc} | m \in \mathbb{Z}\}$. Vice versa if $\tau_d \in K$ for some $d \in \mathbb{R}$, then write $\frac{d}{c} = \ell + \epsilon$ with $\ell \in \mathbb{Z}$ and $0 \leq \epsilon < 1$. We have $d = \ell c + \epsilon c$ and therefore $\tau_{\epsilon c} = \tau_d \tau_{-\ell c} \in K$. By definition c is the smallest positive number with $\tau_c \in K$, so $0 \leq \epsilon c < c$ implies $\epsilon = 0$. This shows $d = \ell c$ and $\tau_d \in \{\tau_{mc} | m \in \mathbb{Z}\}$, completing the proof. ■

We now present a description of all frieze groups.

V.4.9 Theorem. *All frieze groups are of exactly one of the following 7 types:*

F1. Groups $\{\tau_{mc} | m \in \mathbb{Z}\}$ (for fixed $c > 0$) consisting of translations;

F2. Groups $\{\gamma_c^m | m \in \mathbb{Z}\}$ (fixed $c > 0$) consisting of glide reflections and translations;

F3. Groups $\{\tau_{mc} | m \in \mathbb{Z}\} \cup \{\gamma_{mc} | m \in \mathbb{Z}\}$ (fixed $c > 0$) consisting of glide reflections and translations, including the reflection $\gamma = \gamma_0$;

F4. Groups $\{\tau_{mc} | m \in \mathbb{Z}\} \cup \{\rho_{u-mc} | m \in \mathbb{Z}\}$ (fixed $c > 0$ and fixed u) consisting of translations and reflections in vertical lines;

F5. Groups $\{\tau_{mc} | m \in \mathbb{Z}\} \cup \{\pi_{u-mc} | m \in \mathbb{Z}\}$ (fixed $c > 0$ and fixed u) consisting of translations and point reflections;

F6. Groups $\{\gamma_{c/2}^n | n \in \mathbb{Z}\} \cup \{\rho_u \gamma_{c/2}^n | n \in \mathbb{Z}\}$ (fixed $c > 0$ and fixed u) consisting of translations, glide reflections, reflections in vertical lines, and point reflections, not containing the glide reflection $\gamma = \gamma_0$.

F7. Groups $\{\tau_{mc} | m \in \mathbb{Z}\} \cup \{\rho_{u+mc} | m \in \mathbb{Z}\} \cup \{\gamma_{mc} | m \in \mathbb{Z}\} \cup \{\pi_{u+mc} | m \in \mathbb{Z}\}$ (fixed $c > 0$ and fixed u) consisting of translations, glide reflections, reflections in vertical lines, and point reflections, including the glide reflection $\gamma = \gamma_0$.

Proof. Let $F \subset G_S$ be a frieze group. We use the restriction to F of the homomorphism $\varphi : G_S \to H = \{\mathrm{id}, \gamma, \rho, \pi\}$ which we will (also) denote $\varphi : F \to H$. Its kernel equals $F \cap T$ and in the proof of Theorem V.4.8 we saw that this kernel consists of the translations τ_d^m for some fixed $d > 0$ and all $m \in \mathbb{Z}$. The image $\varphi(F) \subset H$ is by Theorem III.3.4 a subgroup of H. By discussing the possibilities for $\varphi(F)$ one by one, all possible F's will arise. Lagrange's theorem III.2.8 asserts that $\#\varphi(F) | \#H$, so $\#\varphi(F)$ equals $1, 2$, or 4. In the latter case $\varphi(F) = H$, in the first case $\varphi(F) = \{\mathrm{id}\}$. In all other cases $\varphi(F)$ contains besides the identity id exactly one other element, of order 2. So we have the following possibilities.

1. $\varphi(F) = \{\mathrm{id}\}$. In this case F contains only translations. The proof of Theorem V.4.8 shows that $c > 0$ exists with $F = \{\tau_{mc} | m \in \mathbb{Z}\}$, which is a group isomorphic to $\mathbb{Z}$.
2. $\varphi(F) = \{\mathrm{id}, \gamma\}$; if this happens, then we are in one of the following cases.
   (a) Every glide reflection $\gamma_u = \tau_u \gamma \in F$ has infinite order, i.e., it satisfies $u \neq 0$. Put $K = F \cap T$ which, as above, can also be written as $K = \{\tau_{mc} | m \in \mathbb{Z}\}$ for some $c > 0$. Any $\gamma_u \in F$ yields $\gamma_u^2 = \tau_{2u} \in K$. Hence $u = mc/2$ for some $m \in \mathbb{Z}$. Now write $m = 2q + r$ with $q \in \mathbb{Z}$ and $r \in \{0, 1\}$. Then $\tau_{-qc} \in F$ and therefore also $\tau_{-qc}\gamma_{mc/2} = \gamma_{rc/2} \in F$. We conclude that $r = 1$ since otherwise F would contain γ_0, contrary to the assumption. As a consequence $\gamma_{c/2} \in F$ and moreover the glide reflections in F are precisely all $\gamma_{mc/2}$ with m an odd integer. Observe that $\gamma_{c/2}^n$ equals the translation τ_{mc} in case $n = 2m$ is even, and equals the glide reflection $\gamma_{mc/2}$ in case $n = 2m - 1$ is odd. This means $F = \{\gamma_{c/2}^n | n \in \mathbb{Z}\} = \{\tau_{mc} | m \in \mathbb{Z}\} \cup \{\gamma_{c/2}\tau_{mc} | m \in \mathbb{Z}\}$. The map $n \mapsto \gamma_{c/2}^n$ yields $\mathbb{Z} \cong F$.
   (b) F contains a glide reflection of finite order, i.e., $\gamma \in F$. As before, take $c > 0$ such that $F \cap T = \{\tau_{mc} | m \in \mathbb{Z}\}$. If γ_u is any glide reflection in F, then also $\gamma\gamma_u = \tau_u$ is in F which means $u = mc$ for some integer m. The converse holds as well: given $m \in \mathbb{Z}$ we have $\gamma_{mc} = \gamma\tau_{mc} \in F$. So
   $$F = \{\tau_{mc} | \, m \in \mathbb{Z}\} \cup \{\gamma_{mc} | \, m \in \mathbb{Z}\},$$
   and $(m, \overline{0}) \mapsto \tau_{mc}$ and $(m, \overline{1}) \mapsto \gamma_{mc}$ defines an isomorphism $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) \cong F$.
3. $\varphi(F) = \{\mathrm{id}, \rho\}$. Again, there is a $c > 0$ such that $F \cap T = \{\tau_{mc} | m \in \mathbb{Z}\}$. Take any reflection $\rho_u \in F$, then $\rho_u \tau_c \rho_u = \tau_{-c}$; in particular it follows that F is not abelian. If also $\rho_v \in F$ then $\rho_u \rho_v = \tau_{u-v} \in F$ so $u - v = mc$ and $\rho_v = \rho_u \tau_{mc} = \rho_{u-mc}$. Clearly all products $\rho_u \tau_{mc}$ with $m \in \mathbb{Z}$ are in F, so
   $$F = \{\tau_{mc} | \, m \in \mathbb{Z}\} \cup \{\rho_{u-mc} | \, m \in \mathbb{Z}\}$$
   which indeed defines an infinite discrete group.
4. $\varphi(F) = \{\mathrm{id}, \pi\}$. This case is almost identical to the one above: let $c > 0$ such that $F \cap T = \{\tau_{mc} | m \in \mathbb{Z}\}$. Take any point reflection $\pi_u \in F$, then $\pi_u \tau_c \pi_u = \tau_{-c}$; in particular it follows that F is not abelian. If also $\pi_v \in F$ then $\pi_u \pi_v = \tau_{u-v} \in F$ so $u - v = mc$ and $\pi_v = \pi_u \tau_{mc} = \pi_{u-mc}$. Clearly all products $\pi_u \tau_{mc}$ with $m \in \mathbb{Z}$ are in F, so
   $$F = \{\tau_{mc} | \, m \in \mathbb{Z}\} \cup \{\pi_{u-mc} | \, m \in \mathbb{Z}\}$$
   which indeed defines an infinite discrete group.
5. $\varphi(F) = H = \{\mathrm{id}, \rho, \gamma, \pi\}$. In this case the elements of F that are either translations or reflections in a vertical line form a subgroup $F' \subset F$. In part 3. above it such groups F' were described: it is non-abelian and its elements are given in terms of two elements τ_c and ρ_u with $\rho_u \tau_c \rho_u = \tau_c^{-1}$. The remainder of the argument is in the spirit of Case 2. above:

(a) Assume $\gamma = \gamma_0 \notin F$. Let $\gamma_v \in F$ be a glide reflection. Then $\gamma_v^2 = \tau_{2v} \in F$. Exactly as in 2(a) above, the translations and glide reflections in F are precisely the powers $\gamma_{c/2}^n$ of $\gamma_{c/2} \in F$. Moreover $\rho_u \gamma_{c/2} \rho_u = \gamma_{c/2}^{-1}$. We claim that all elements of F can be expressed in terms of ρ_u and $\gamma_{c/2}$. This was shown above for the translations, glide reflections, and the reflections in vertical lines. If $\pi_w \in F$ is a point reflection then $\pi_w = \rho_u \rho_u \pi_w = \rho_u \gamma_{u-w}$, where $\gamma_{u-w} = \rho_u^{-1} \pi_w \in F$ is a power of $\gamma_{c/2}$, which shows the claim in the remaining case. In fact, we find

$$F = \{\gamma_{c/2}^n | \, n \in \mathbb{Z}\} \cup \{\rho_u \gamma_{c/2}^n | n \in \mathbb{Z}\}$$

which indeed defines an infinite discrete subgroup of G_S.

(b) Assume $\gamma = \gamma_0 \in F$. Note that $\gamma g = g\gamma$ for every $g \in G_S$, hence this holds in particular for all $g \in F$. Take $F' \subset F$ the subgroup of F consisting of all translations and all reflections across vertical lines. We know from the third part of this proof that $F' = \{\tau_{mc} | \, m \in \mathbb{Z}\} \cup \{\rho_{u-mc} | \, m \in \mathbb{Z}\}$ for some $c > 0$. As before, one shows that F is the disjoint union $F' \cup \gamma F'$: clearly this union is in F and the sets $F', \gamma F'$ are disjoint. Moreover if $g \in F$, then it is in F' in case g is either a translation or a reflection in a vertical line. In the case g that is a point reflection or a glide reflection then $\gamma g \in F'$ and hence $g = \gamma\gamma g \in \gamma F'$. We conclude

$$F = \{\tau_{mc} | \, m \in \mathbb{Z}\} \cup \{\rho_{u+mc} | m \in \mathbb{Z}\} \cup \{\gamma_{mc} | \, m \in \mathbb{Z}\} \cup \{\pi_{u+mc} | \, m \in \mathbb{Z}\}$$

which indeed is an infinite discrete subgroup of G_S.

This completes the description of all frieze groups.

■

V.4.10 Remark. Observe that the group(s) encountered in the cases 3. and 4. and 5(a) above, are very reminiscent of the dihedral groups whose properties are listed in Theorem V.3.4. Namely, there is an element $\sigma \in \{\rho_u, \pi_v\}$ or order 2, and an element $r \in \{\tau_c, \rho_{c/2}\}$ of infinite order (in the case of the dihedral group D_n instead a rotation of order n is taken), and $\sigma r \sigma = r^{-1}$.

V.4.11 Remark. At first sight the groups appearing in the cases 5(a) and 5(b) of the preceding argument may look exactly the same. However, the groups are not even isomorphic. Indeed, starting from a group $F_4 = \{\tau_{mc} | \, m \in \mathbb{Z}\} \cup \{\rho_{u-mc} | \, m \in \mathbb{Z}\}$ (case 3 of the theorem) one obtains a group as in 5(a) as $F_6 = F_4 \cup \gamma_v F_4$ in which $v \notin \mathbb{Z} \cdot c$. A group as in 5(b) is given by $F_7 = F_4 \cup \gamma F_4$. The center (compare Exercise III.4(7)) $\mathcal{Z}(F_6)$ equals {id} whereas $\mathcal{Z}(F_7) = \{\mathrm{id}, \gamma\}$, and isomorphic groups have isomorphic centers as well.

As abstract groups different types of frieze groups can be isomorphic, as we saw in the proof above and in the Remarks following the proof. Namely, the groups in F1 and F2 are all isomorphic to $\mathbb{Z}$, although in F1 a group contains only translations whereas in F2 also glide reflections occur. In a similar way the groups in F4, F5, and F6 are all isomorphic, although the different types contain different kinds of symmetries of the strip. In total we found 7 types of frieze groups, but only 4 different groups up to isomorphism. Only the groups in F1, F2, and F3 are abelian. Those in F3 contain an element of order 2, hence they are not isomorphic to a group in F1 or F2. All groups in F4, F5, F6, F7 are non-abelian. We already saw that three of these cases yield isomorphic groups, and also that the case F7 leads to groups not isomorphic to the one in the other cases.

We conclude by presenting examples of drawings on the strip S which have as symmetry group a given frieze group. Obviously we cannot extend the drawing unboundedly to the left or to the right, but it should be obvious how this is done.

Type F1:

⋯b b b b b b b b b b b b b b b b b b b b b b b b b b b b b b b b⋯

Type F2:

⋯b p b p b p b p b p b p b p b p b p b p b p b p b p b p b p b p⋯

Type F3:

⋯C C C C C C C C C C C C C C C C C C C C C C C C C C C C⋯

Type F4:

⋯b d b d b d b d b d b d b d b d b d b d b d b d b d b d b d b d⋯

Type F5:

⋯b q b q b q b q b q b q b q b q b q b q b q b q b q b q b q b q⋯

Type F6:

⋯b p q d b p q d b p q d b p q d b p q d b p q d b p q d b p q d⋯

Type F7:

⋯x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x x⋯

Many examples of patterns with a frieze group as group of symmetries can be found on the internet, including their appearance in decorative art. The websites `http://www.maa.org/sites/default/files/images/upload_library/4/vol1/architecture/Math/seven.html`
and
`https://en.wikipedia.org/wiki/Frieze_group`
provide a first impression.

V.5 Automorphisms of a graph

We will consider the simplest and easiest type of graph here. In particular we restrict ourselves to finite graphs with at most one edge between its vertices. Moreover, we do not prescribe a direction for the edges of our graphs. In more advanced graph theory the graphs we use, are called "finite simple undirected graphs"; our terminology will be shorter:

V.5.1 Definition. A graph Γ is a pair (V,E), with V a nonempty finite set (the 'vertices' of the graph), and E a (possibly empty) set consisting of subsets $\{a,b\} \subset V$ (the 'edges' of the graph).

V.5.2 Remark. A graph is usually presented as a picture: we draw its vertices as points, and we connect vertices a,b by a line segment (or by a loop in case $a = b$) precisely in the case $\{a,b\}$ is in the set of edges of the graph. In many examples such a picture is only possible if we allow some of the line segments to intersect. By emphasizing the actual vertices of the graph, one can make sure that no confusion with the intersection points of line segments arises.

To a graph one associates a finite group as follows.

V.5.3 Definition. An automorphism of a graph $\Gamma = (V, E)$ is a permutation σ on its set of vertices V, with the property that for all $\{a, b\} \in E$ also $\{\sigma(a), \sigma(b)\} \in E$.

The set consisting of all automorphisms of Γ is denoted $\mathrm{Aut}(\Gamma)$.

V.5.4 Theorem. *For a graph Γ with n vertices,* $\mathrm{Aut}(\Gamma)$ *is a subgroup of S_n.*

Proof. Enumerate the vertices of the graph Γ as $1, 2, \ldots, n$. It is clear that any $\sigma \in \mathrm{Aut}(\Gamma)$ corresponds to a permutation in S_n, so $\mathrm{Aut}(\Gamma) \subset S_n$. We now show that this subset is a subgroup. The identity is contained in it. If $\sigma \in \mathrm{Aut}(\Gamma)$, then σ maps elements of E to elements of E as $\sigma(\{i, j\}) := \{\sigma(i), \sigma(j)\}$. This yields an injective map from E to E, and since E is finite this map is surjective as well. This means that in case $\sigma(k) = i$ and $\sigma(\ell) = j$ and $\{i, j\} \in E$, then also $\{k, \ell\} \in E$. The definition of $\mathrm{Aut}(\Gamma)$ therefore shows that if $\sigma \in \mathrm{Aut}(\Gamma)$, then $\sigma^{-1} \in \mathrm{Aut}(\Gamma)$ as well. To prove that a product of elements in $\mathrm{Aut}(\Gamma)$ yields an element of $\mathrm{Aut}(\Gamma)$ is much simpler so we leave it for the reader. This completes the proof. ∎

V.5.5 Example. The *complete* graph Γ_n on n vertices is by definition the graph consisting of n vertices $1, 2, \ldots, n$, and edges all $\{i, j\}$ with $1 \leq i \leq j \leq n$. In this example the requirement that an automorphism sends edges to edges yields no restriction, hence $\mathrm{Aut}(\Gamma_n) = S_n$. ▬

V.5.6 Example. Enumerate the vertices of a regular n-gon as $1, 2, \ldots, n$ (say, counter clockwise). Regard this n-gon as a graph F_n, so with vertices $1, 2, \ldots n$ and edges $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{n, 1\}$. Then $\mathrm{Aut}(F_n) \cong D_n$: namely, every symmetry of the regular n-gon can be considered as an element of $\mathrm{Aut}(F_n)$, so $D_n \subset \mathrm{Aut}(F_n)$. Vice versa, if $\tau \in \mathrm{Aut}(F_n)$ sends the vertex 1 to i, then 2 is mapped to one of the two neighbours of i, and this determines τ uniquely. As a consequence, at most $n \cdot 2 = 2n$ possible τ exist. We found this number of elements in the subset D_n, so indeed $\mathrm{Aut}(F_n) \cong D_n$. ▬

V.6 Exercises

1. Show that $D_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $D_3 \cong S_3$.
2. By considering the elements of D_n as permutations on the vertices of a regular n-gon one obtains a map from D_n to S_n. Verify that this map is an injective homomorphism. Which elements of D_n are mapped to even permutations?
3. Take $v \in \mathbb{R}^n$ and denote translation by v as τ_v. Let $a \in \mathbb{R}$ with $a \neq 0$.
   (a) Verify that $a\tau_v \frac{1}{a}$ is again a translation.
   (b) Show that if φ is any isometry on $\mathbb{R}^n$, then so is $a\varphi\frac{1}{a}$.
4. Let F_7 be a frieze group of type F7, and let $F_4 \subset F_7$ be its subgroup consisting of all translations and all reflections in vertical lines.
   (a) Show that every element $g \in F_7$ can be written in a unique way as $g = ab$ with $a \in \{\mathrm{id}, \gamma\}$ and $b \in F_4$.
   (b) Prove that $(\overline{n}, b) \mapsto \gamma^n b$ defines an isomorphism $(\mathbb{Z}/2\mathbb{Z}) \times F_4 \cong F_7$.
5. Find the type (according to Theorem V.4.9) of the symmetry group of each of the following infinite patterns:
   (a) ⋯s s s s s s s s s s s s s s s s s s s s s s s s s⋯
   (b) ⋯v v v v v v v v v v v v v v v v v v v v v v v v v⋯
   (c) ⋯r r r r r r r r r r r r r r r r r r r r r r r r r⋯
6. Verify that exactly 20 distinct graphs with exactly 3 vertices exist, and that none of them has A_3 as automorphism group.
7. Determine the number of automorphisms and the group $\mathrm{Aut}(H)$, with H the graph consisting of 6 vertices and 5 edges, drawn as the capital letter 'H'.
8. A cube can be considered as a graph by taking its 8 vertices as vertices and its 12 sides as edges. Determine the automorphism group of this graph.
9. The groups considered in this chapter all consist of bijections on a certain set, where the bijections are required to preserve some additional structure on the set. A natural additional example is to take as the set some group G, and to consider the bijections $\tau : G \to G$ which preserve the group structure:

$$\mathrm{Aut}(G) := \{\tau \in S_G \mid \tau(gh) = \tau(g)\tau(h)\}.$$

   (a) Determine $\mathrm{Aut}(\mathbb{Z}/4\mathbb{Z})$.
   (b) Show that $\mathrm{Aut}(G)$ is a group for any group G.
   (c) Show that $G \to \mathrm{Aut}(G)$ defined by $g \mapsto \gamma_g$, with γ_g defined by $\gamma_g(h) = ghg^{-1}$, indeed maps G to $\mathrm{Aut}(G)$, and that it is a group homomorphism from G to $\mathrm{Aut}(G)$.

VI CONJUGATION, INDEX, ACTIONS, AND SYLOW THEORY

After two chapters in which we focused on *examples* of groups, we now continue the general theory started in Chapter III. In particular we develop additional theory on subgroups of (mostly finite) groups; this is useful, for instance, when trying to classify all groups of some given order up to isomorphism. We also discuss multiplication on the left or on the right by a fixed element in more detail. It turns out that this will allow us to partition a given group into certain subsets, its conjugacy classes. In fact this provides an example of an important general notion, namely 'group actions'.

VI.1 Conjugation

Consider an arbitrary group G and fix elements $a, b \in G$. The maps λ_a and ρ_a from G to G defined as 'multiplication on the left by a' (so $\lambda_a(x) = ax$) and 'multiplication on the right by b' ($\rho_a(x) = xb$) are bijections. Their composition given by $x \mapsto axb$, is therefore bijective as well. Since the most important maps between groups are homomorphism, it natural to ask when this composition is a homomorphism. In general this may not be the case, because a homomorphism maps the unit element to the unit element by Theorem III.3.3, and our bijection maps $e \in G$ to $aeb = ab$. This equals the unit element e if and only if b is the inverse of a, so $b = a^{-1}$. We now study this case in more detail.

VI.1.1 Definition. If G is a group and $a \in G$, then the bijection $\gamma_a : G \to G$ given by $\gamma_a(x) = axa^{-1}$ is called the *conjugation* by a.

VI.1.2 Theorem. *Let G be a group and let $a, b \in G$.*

1. *The conjugation γ_a by a is an isomorphism.*
2. *The conjugations γ_a, γ_b satisfy $\gamma_a\gamma_b = \gamma_{ab}$.*
3. *The inverse of γ_a is $\gamma_{a^{-1}}$.*
4. *If H is a subgroup of G, then so is $\gamma_a(H) = aHa^{-1}$, and $H \cong aHa^{-1}$.*

Proof. 1. For $x, y \in G$ we have

$$\gamma_a(xy) = axya^{-1} = axa^{-1}aya^{-1} = \gamma_a(x)\gamma_a(y).$$

So γ_a is a homomorphism. We already observed that γ_a is bijective, so it is an isomorphism.
2. If $x \in G$, then

$$\gamma_a\gamma_b(x) = \gamma_a(bxb^{-1}) = abxb^{-1}a^{-1} = (ab)x(ab)^{-1} = \gamma_{ab}(x).$$

In other words, $\gamma_a\gamma_b = \gamma_{ab}$.
3. It follows from 2. that $\gamma_a\gamma_{a^{-1}} = \gamma_e = \gamma_{a^{-1}}\gamma_a$. Every $x \in G$ satisfies $\gamma_e(x) = exe^{-1} = x$, so $\gamma_e = \mathrm{id}$. This shows the assertion.
4. Since γ_a is a homomorphism and H is a group, $\gamma_a(H)$ is a group as well. The map γ_a is injective, so this also holds for its restriction to H. This restriction has image $\gamma_a(H)$ by definition, hence $H \cong \gamma_a(H)$. ∎

VI.1.3 Example. If G is a commutative group, then conjugation by an arbitrary element of G is the identity map. So conjugation can only be of interest for non-abelian groups. ▬

VI.1.4 Example. In linear algebra conjugating matrices plays a major role when changing the basis of a vector space. ▬

VI.1.5 Definition. Two elements x, y in a group G are called *conjugate* if a conjugation γ_a for some $a \in G$ exists with $\gamma_a(x) = y$.

The *conjugacy class* of $x \in G$ defined as the subset of G given by

$$C_x = \left\{ y \in G \mid \text{there exists } a \in G \text{ with } \gamma_a(x) = y \right\}.$$

VI.1.6 Example. In an abelian group G every $x \in G$ satisfies $C_x = \{x\}$.

In S_3 the cycles (1 2) and (1 2 3) are *not* conjugate. To se this, note that all $\tau \in S_3$ satisfy $\tau(1\,2)\tau^{-1} = (\tau(1)\,\tau(2))$ and $\tau(1\,2\,3)\tau^{-1} = (\tau(1)\,\tau(2)\,\tau(3))$ by Lemma IV.3.6. It follows that (1 2) is conjugate to every 2-cycle, and (1 2 3) to every 3-cycle, but the two given cycles are not conjugate. ▬

VI.1.7 Example. The theory of Jordan normal forms in linear algebra shows that two matrices $A, B \in \mathrm{GL}_n(\mathbb{C})$ are conjugate if and only if they have the same Jordan form. For example, $\left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$ en $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ are *not* conjugate, but $\left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are. ▬

VI.1.8 Theorem. Let G be a group and let $x, y, z \in G$.

1. *The element x is conjugate to itself, so $x \in C_x$.*
2. *If x is conjugate to y, then also y is conjugate to x (so $x \in C_y$ implies $y \in C_x$).*
3. *If $x \in C_y$ and $y \in C_z$, then $x \in C_z$.*

Proof. 1. We have $x = \gamma_e(x)$, so $x \in C_x$ for all $x \in G$.
2. $x, y \in G$ satisfy $x = \gamma_a(y)$ precisely when $y = \gamma_{a^{-1}}(x)$. This shows the assertion.
3. Suppose that $a, b \in G$ exist with $\gamma_a(y) = x$ and $\gamma_b(z) = y$. It now follows that $\gamma_{ab}(z) = \gamma_a\gamma_b(z) = \gamma_a(y) = x$, so $x \in C_z$. ∎

VI.1.9 Corollary. Every group G is the disjoint union of conjugacy classes. In other words, every element of G lies in some C_x, and if there is an element in both C_x and C_y, then $C_x = C_y$.

Proof. Every $a \in G$ lies in C_a. If $a \in C_x$ and $a \in C_y$, then $c, d \in G$ exist with $a = \gamma_c(x)$ and $a = \gamma_d(y)$. Any $z \in C_x$ can therefore be written as $z = \gamma_f(x) = \gamma_{fc^{-1}d}(y)$, so $z \in C_y$. The same argument with x, y interchanged shows $C_y \subset C_x$. So $C_x = C_y$. ∎

VI.1.10 Remark. Theorem VI.1.8 says that 'being conjugate' is an equivalence relation, compare Lemma II.1.2. Corollary VI.1.9 says that one can partition a group G with respect to this relation. This is in fact true for any equivalence relation.

VI.1.11 Example. We write S_n as a disjoint union of conjugacy classes. Take $\sigma \in S_n$. Write σ as a product of disjoint cycles:

$$\sigma = (a_1 \dots a_{\ell_1})(a_{\ell_1+1} \dots a_{\ell_2}) \dots (a_{\ell_{s-1}+1} \dots a_{\ell_s}).$$

A permutation τ sending each of the a_i to i (and the remaining integers in $\{1,\dots,n\}$ bijectively to $\{\ell_s+1,\dots,n\}$) yields

$$\tau\sigma\tau^{-1} = (1\,2 \dots \ell_1)(\ell_1+1 \dots \ell_2) \dots (\ell_{s-1}+1 \dots \ell_s).$$

We conclude that all products of disjoint $\ell_1, \ell_2-\ell_1, \dots, \ell_s-\ell_{s-1}$-cycles are conjugate. The conjugacy class only depends on the set $\{\ell_1, \ell_2-\ell_1, \dots, \ell_s-\ell_{s-1}\}$ (the "*cycle type*").. In particular the number of pairwise different conjugacy classes equals the number of *partitions* of n; this is the number of presentations $n = \sum n_i$ with n_i positive integers, where the order of n_i's is not taken into account. The number of partitions is denoted $p(n)$. So $p(2) = 2$ since $2 = 2$ and $2 = 1+1$, and $p(4) = 5$ $(4, 3+1, 2+2, 2+1+1, 1+1+1+1)$. ∎

VI.1.12 Example. Determining the conjugacy classes in the alternating group A_n is considerably more involved than the case of S_n. We consider $n \le 5$ here. First note that A_n is commutative for $n \le 3$, so in these cases $C_\sigma = \{\sigma\}$ for all $\sigma \in A_n$.

The group A_4 consists of (1), three products of two disjoint 2-cycles, and eight 3-cycles. Write $\{3,4\} = \{a,b\}$, then $\tau = (2\,a\,b) \in A_4$ satisfies $\tau(1\,2)(3\,4)\tau^{-1} = (1\,a)(b\,2)$. So all products of two disjoint 2-cycles in A_4 are conjugate. Conjugating (1 2 3) by all 12 elements in A_4 one finds $C_{(1\,2\,3)} = \{(1\ 2\ 3), (1\ 3\ 4), (1\ 4\ 2), (2\ 4\ 3)\}$. The remaining four 3-cycles form a conjugacy class as well.

The group A_5 consists of 3-cycles, 5-cycles, products of 2 disjoint 2-cycles, and the identity. All 3-cycles form one conjugacy class. To see this, note that if $\tau \in S_5$ is a permutation sending each a_i to i, and if $\sigma = (a_1\,a_2\,a_3)$, then $\tau\sigma\tau^{-1} = (1\,2\,3)$. However, we need to check whether there is an *even* τ sending each a_i to i. Choose any $\tau \in S_5$ with this property. Multiplying τ on the left by the 2-cycle (4 5) does not change the property; this shows that we may indeed assume τ to be even. The products of two disjoint 2-cycles are all conjugate as well. Indeed, such a product σ fixes a unique positive integer $i \le 5$, and any $\tau\sigma\tau^{-1}$ then fixes $\tau(i)$ (and no other positive integer ≤ 5). Using a suitable even τ then shows that σ is conjugate to a product fixing 5. So we find precisely the products of two disjoint 2-cycles that were discussed when finding the conjugacy classes in A_4. As we saw there, they are all conjugate.

It remains to look at the 5-cycles. A 5-cycle σ can be written as $\sigma = (1\,a\,b\,c\,d)$, so there are 24 of them. We have $\tau\sigma\tau^{-1} = \sigma$ if and only if

$$(\tau(1)\,\tau(a)\,\tau(b)\,\tau(c)\,\tau(d)) = (1\,a\,b\,c\,d),$$

which means that τ must be a power of σ. The powers of σ form a subgroup $H := \langle\sigma\rangle$ of A_n consisting of 5 elements. Write A_n as a disjoint union of subsets $H\pi$, with $\pi \in A_n$. If $\tau \in H\pi$, then $\tau\sigma\tau^{-1} = \pi\sigma\pi^{-1}$. Moreover, for $\pi_1, \pi_2 \in A_5$ we have $\pi_1\sigma\pi_1^{-1} = \pi_2\sigma\pi_2^{-1}$ if and only if $\pi_2^{-1}\pi_1 \in H$, i.e., $\pi_1 \in H\pi_2$. So for a fixed 5-cycle σ there are as many pairwise distinct elements $\tau\sigma\tau^{-1}$ as there are distinct sets $H\pi$. The number of these is $\#A_5/\#H = 12$. We conclude that the set of 5-cycles consists of two conjugacy classes, each containing 12 elements. In total we find 5 conjugacy classes, consisting of $1, 20, 15, 12$, and 12 elements, respectively. ∎

The example above proves a result for A_5 which is true for general groups.

VI.1.13 Theorem. *If G is a group and $a \in G$, then $N(a) := \{x \in G \mid \gamma_x(a) = a\}$ is a subgroup of G. If G is finite, then*

$$\#G = \#N(a) \cdot \#C_a.$$

Proof. Using $\gamma_e = \mathrm{id}$ and $\gamma_{x^{-1}} = \gamma_x^{-1}$ and $\gamma_{xy} = \gamma_x\gamma_y$, we see that $N(a)$ is a subgroup of G. The proof of Theorem III.2.8 shows that G is a disjoint union of subsets $g_iN(a)$, for pairwise distinct $g_i \in G$. Suppose that G is finite. Each of the subsets $g_iN(a)$ has $\#N(a)$ elements, so the proof is complete if we can show that the number of g_i's equals $\#C_a$, i.e. that a bijection $\{g_1,\ldots,g_i,\ldots\} \to C_a$ exists.

We claim that the map defined by $g_i \mapsto x_i := \gamma_{g_i}(a) \in C_a$ is bijective. Namely, if $x \in C_a$, then $x = \gamma_g(a)$ for some $g \in G$. Therefore $g \in g_iN(a)$ for some i, hence $g = g_ih$ with $h \in N(a)$, and we deduce $x = \gamma_g(a) = \gamma_{g_i}\gamma_h(a) = \gamma_{g_i}(a) = x_i$. So the map is surjective. If $x_i = x_j$, then $g_j^{-1}g_iag_i^{-1}g_j = a$ hence $g_j^{-1}g_i \in N(a)$. It follows that $g_iN(a) = g_jN(a)$, thus $g_i = g_j$. Hence the map is injective as well, completing the proof. ∎

As an application we determine some conjugacy classes in A_n for $n \geqslant 5$.

VI.1.14 Theorem. *Let $n \geq 5$.*

1. *In A_n all 3-cycles are conjugate.*
2. *In A_n all products of two disjoint 2-cycles are conjugate.*

Proof. Let $\sigma = (1\,2\,3) \in A_n$. By definition, the subgroup $N(\sigma) \leqslant A_n$ consists of all even permutations τ satisfying $\tau\sigma\tau^{-1} = \sigma$, which means $(\tau(1)\,\tau(2)\,\tau(3)) = (1\,2\,3)$. Hence τ is a power of $(1\,2\,3)$ times an even permutation on $\{4,5,\ldots,n\}$, so we find $\#N(\sigma) = 3\cdot(n-3)!/2$. Note that here the condition $n \geq 5$ is used. Theorem VI.1.13 implies $\#C_\sigma = (n!/2)/(3\cdot(n-3)!/2) = 2\binom{n}{3}$. This is equal to the number of 3-cycles in A_n, hence we conclude that all 3-cycles are conjugate, because C_σ consists of 3-cycles.

The same idea can be adapted to the case of a product of two disjoint 2-cycles. We leave it as a useful exercise to the reader. ∎

VI.1.15 Remark. We sketch an alternative proof. Take a 3-cycle $(a\ b\ c)$. Choose a permutation τ with $\tau(1) = a, \tau(2) = b$, and $\tau(3) = c$. Conjugation by τ and by $\tau\cdot(4\,5)$ both map $(1\,2\,3)$ to $(a\ b\ c)$. Since one of τ and $\tau\cdot(4\,5)$ is even, $(1\,2\,3)$ and $(a\ b\ c)$ are conjugate in A_n.

The case of products of two disjoint 2-cycles can be treated similarly; the reader should try to verify this.

VI.2 Index

Let G be a group and let $H \leqslant G$ a subgroup. Recall that all sets of the form gH and Hg are bijective, since multiplication by g on the left (on the right, respectively) induces a bijection between H and gH (Hg, respectively). In particular, we have already used that when H is finite, these sets all have the same number of elements. Another important property is the fact that for $g_1, g_2 \in G$ we either have $g_1H = g_2H$ (and this holds precisely when $g_1g_2^{-1} \in H$), or $g_1H \cap g_2H = \varnothing$. The analogous statement holds for the sets Hg_1 and Hg_2. Recall for yourself how these assertions are proven!

VI.2.1 Definition. For H a subgroup of a group G, a *left coset* of H in G is any subset of the form gH, for $g \in G$. The *index* of H in G is defined as the number of disjoint left cosets of H in G. The index is denoted by $[G : H]$. If the index is not finite then we write $[G : H] = \infty$.

The set consisting of all left cosets of H in G is denoted G/H. In other words,

$$G/H := \{gH \,:\, g \in G\}.$$

VI.2.2 Remark. Since 'taking the inverse' $\iota : G \to G$ is a bijection and $\iota(gH) = H\iota(g)$, one may also define the index as the number of disjoint subsets of the form Hg (these subsets are called 'right cosets' of H in G).

Observe that $[G : H] = \#(G/H)$.

VI.2.3 Theorem. *If G is a finite group, then $[G : H]$ is finite for all subgroups H. Moreover, we have*

$$\#G = [G : H] \cdot \#H.$$

Proof. This was already shown in the proof of Theorem III.2.8. ∎

VI.2.4 Example. It is certainly possible that a subgroup of an infinite group G has finite index. For example taking $G = \mathbb{Z}$, the subgroups are the groups $n\mathbb{Z}$ by Example III.2.7. For $n \neq 0$ we have $\mathbb{Z} = n\mathbb{Z} \cup (1 + n\mathbb{Z}) \cup \ldots \cup ((n-1) + n\mathbb{Z})$, so $[\mathbb{Z} : n\mathbb{Z}] = n$. Moreover, $[\mathbb{Z} : 0\mathbb{Z}] = \infty$. ▁■

VI.3 Action, Orbit, Stabilizer

The elements of the permutation group S_n are by definition maps from the set $\{1, 2, \ldots, n\}$ to itself: given $\tau \in S_n$ and $m \in \{1, 2, \ldots, n\}$ then also $\tau(m) \in \{1, 2, \ldots, n\}$. In other words, this defines a map

$$S_n \times \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\} : \qquad (\tau, m) \mapsto \tau(m).$$

Similarly, the group $\mathrm{GL}_2(\mathbb{R})$ consists of maps $\mathbb{R}^2 \to \mathbb{R}^2$. So given $\varphi \in \mathrm{GL}_2(\mathbb{R})$ and $v \in \mathbb{R}^2$, also $\varphi(v) \in \mathbb{R}^2$ and one obtains

$$\mathrm{GL}_2(\mathbb{R}) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \qquad (\varphi, v) \mapsto \varphi(v).$$

These two examples and some properties they share, are formalized in the concept of a 'group action'. This chapter discusses the definition and some basic properties.

The main subject of this section is the following notion.

VI.3.1 Definition. Let G be a group and X a nonempty set. A group action of G on X is a map $G \times X \to X$ which we write as $(g, x) \mapsto gx$, satisfying

A1. $ex = x$ for every $x \in X$ (here $e \in G$ is the unit element);
A2. $(gh)x = g(hx)$ for all $g, h \in G$ and all $x \in X$.

Instead of 'group action of G on X' one also writes 'X is a G-set', or 'G acts on X'.

VI.3.2 Example. The permutation group S_n acts on $X = \{1, 2, \ldots, n\}$ via the rule $(\tau, m) \mapsto \tau m := \tau(m)$, as is readily verified.

Similarly, for $n \in \mathbb{Z}_{\geq 1}$ and G any subgroup of the group $\mathrm{GL}_n(\mathbb{R})$, an action of G on $\mathbb{R}^n$ is defined by $\varphi v := \varphi(v)$ (for all $\varphi \in G$ and all $v \in \mathbb{R}^n$).

The group $\mathbb{Z}$ acts on any group G by $ng := g^n$ (for $n \in \mathbb{Z}, g \in G$). ▁■

Recall (Section IV.1) that if X is a nonempty set, then S_X denotes the group consisting of all bijections $X \to X$ (with composition of maps as group operation, and the identity map as unit element). One can describe a group action of a group G on a set X in terms of G and S_X, as follows.

VI.3.3 Theorem. i. *Given an action of the group G on a set X, the map $f : G \to S_X$ given by $f(g)(x) = gx$ is well-defined, and it is a homomorphism.*

ii. *Vice versa, if $f : G \to S_X$ is any homomorphism, then $gx := f(g)(x)$ (for $g \in G$ and $x \in X$) defines an action of G on X.*

Proof. i.: we first show that for $g \in G$, indeed $f(g)$ is an element of S_X. In other words, $f(g)$ is invertible, namely with inverse $f(g^{-1})$. To see this, take any $x \in X$. Then

$$\big(f(g) \circ f(g^{-1})\big)(x) = f(g)(g^{-1}x) = g(g^{-1}x) = (gg^{-1})x = ex = x,$$

so $f(g) \circ f(g^{-1})$ is the identity map. Similarly

$$\big(f(g^{-1}) \circ f(g)\big)(x) = f(g^{-1})(gx) = g^{-1}(gx) = (g^{-1}g)x = ex = x$$

hence $f(g^{-1}) \circ f(g)$ is the identity map as well. This shows that $f(g) \in S_X$.

Next, we will show that f is a homomorphism of groups, in other words, for $g, h \in G$ we have $f(gh) = f(g) \circ f(h)$. To verify the latter condition, again take $x \in X$. Then

$$(f(g) \circ f(h))(x) = f(g)(hx) = g(hx) = (gh)x = f(gh)(x).$$

This proves i.

ii.: given is any group homomorphism $f : G \to S_X$. We verify that $gx := f(g)(x)$ satisfies the properties A1 and A2.
A1: $ex = f(e)(x) = x$ (since $f(e)$ is the unit element of S_X, which is the identity map).
A2: $(gh)x = f(gh)(x) = (f(g) \circ f(h))(x) = f(g)(f(h)(x)) = g(hx)$, finishing the proof. ∎

VI.3.4 Example. The dihedral group D_4 acts on the set X consisting of the 4 vertices of the regular 4-gon (the square). By Theorem IV.1.3 and Theorem VI.3.3 i, this defines a homomorphism $f : D_4 \to S_X \cong S_4$. For a suitable numbering of the vertices, the image of f consists of the permutations (1), (13), (24), (14)(23), (12)(34), (13)(24), (1234), (1432).

The group D_4 also acts on the set Y consisting of the two diagonals of the square. This defines a homomorphism $D_4 \to S_2$.

One can a a straightforward way generalize this example (replace 'square' by $2n$-gon and 'diagonals' by the set of lines passing through the center of the $2n$-gon and containing 2 of its vertices). This results in a homomorphism $D_{2n} \to S_n$. ∎

VI.3.5 Definition. Let the group G act on the set X. and take $x \in X$.

- The *stabilizer* of x in G, denoted by G_x or by $\mathrm{Stab}_G(x)$, is

$$G_x := \{g \in G \,:\, gx = x\} \subseteq G.$$

- The *orbit* of x under G, denoted by Gx, is

$$Gx := \{gx \,:\, g \in G\} \subseteq X.$$

- The action of G on X is called *faithful* if for every pair $g, h \in G$ with $g \neq h$ there exist $y \in X$ with $gy \neq hy$.
- The action of G on X is called *transitive* if for every pair $x_1, x_2 \in X$ there exists $g \in G$ with $gx_1 = x_2$.
- The element $x \in X$ is called a *fixpoint* of G if $Gx = \{x\}$, in other words, if $gx = x$ for every $g \in G$. The set of all fixpoints in X is denoted X^G, so

$$X^G := \{y \in X \,:\, gy = y \text{ for all } g \in G\}.$$

- The action of G on X is called fixpoint free, if there are no fixpoints.

VI.3.6 Example. Let $G = \mathrm{Isom}(\mathbb{R}^2)$ be the group consisting of all distance preserving maps from $\mathbb{R}^2$ to itself (see Section V.2). Then G acts on $X = \mathbb{R}^2$. The stabilizer $G_{(0,0)}$ of the origin is, as we showed in Theorem V.2.3, the subgroup $O(2) \subset \mathrm{Isom}(\mathbb{R}^2)$. The orbit of $(0,0)$ is $G(0,0) = \mathbb{R}^2$: indeed, is $v \in \mathbb{R}^2$, then the translation over v is an element of G which we will denote by g. Then $g(0,0) = v$, so $v \in G(0,0)$ which shows that $\mathbb{R}^2 = G(0,0)$.

The action of G on X is clearly faithful: if two maps $g_1, g_2 \in \mathrm{Isom}(\mathbb{R}^2)$ are different, this means by definition that $g_1(x) \neq g_2(x)$ for some $x \in \mathbb{R}^2$. The action is transitive as well: given $x_1, x_2 \in \mathbb{R}^2$, the translation over $x_2 - x_1$ maps x_1 to x_2. Moreover, the action of $\mathrm{Isom}(\mathbb{R}^2)$ on $\mathbb{R}^2$ has no fixpoints, since, for example, the group contains the translation τ over a point $x \neq (0,0)$ in $\mathbb{R}^2$ and $\tau(y) = y + x \neq y$ for every $y \in \mathbb{R}^2$. In other words, the action is fixpoint free, $X^G = \varnothing$. ∎

We now list some basic properties of the notions Definition VI.3.5 introduces.

VI.3.7 Theorem. *Let G be a group and let X be a G-set. Denote by $f : G \to S_X$ the homomorphism described in Theorem VI.3.3. Then:*

i. *For any $x \in X$, the stabilizer G_x is a subgroup of G.*
ii. *The action of G on X is faithful $\iff$ The map f is injective.*
iii. *G acts transitively on $X \iff Gx = X$ for some $x \in X \iff Gx = X$ for all $x \in X$.*
iv. *For $x, y \in X$ one has $Gx = Gy \iff y \in Gx$, and $Gx \cap Gy = \varnothing \iff y \notin Gx$.*
v. *For $x \in X$ and $g \in G$, one has $G_{gx} = gG_xg^{-1}$. In other words, the conjugation $\gamma_g : G \to G$ (see §VI.1) restricts to an isomorphism $\gamma_g : G_x \cong G_{gx}$.*

Proof. i.: $e \in G_x$, because $ex = x$ by definition of a group action. If $g, h \in G_x$, this means $gx = x = hx$, and therefore $(gh)x = g(hx) = gx = x$, showing that also $gh \in G_x$. Finally, if $g \in G_x$, then $g^{-1}x = g^{-1}(gx) = (g^{-1}g)x = ex = x$, so $g^{-1} \in G_x$. Hence G_x satisfies the conditions for being a subgroup of G (see Theorem III.2.3).
ii.: $\Rightarrow$: we will show that $\ker(f) = \{e\}$, which means by Theorem III.3.6 that f is injective. Take $g \in G$ with $g \neq e$. Since G acts faithfully on X, some $x \in X$ exists with $gx \neq ex$, so $gx \neq x$. So $f(g) \in S_X$ is not the identity map, because $f(g)(x) = gx \neq x$. This shows f is injective.
$\Leftarrow$: given $g \neq h$ in G, injectivity of f means that $f(g) \neq f(h)$. In other words, $x \in X$ exists with $gx = f(g)(x) \neq f(h)(x) = hx$, so G acts faithfully on X.
iii.: suppose G acts transitively on X, and take any $x \in X$. Given $y \in X$, the action being transitive means that $y = gx$ for some $g \in G$. Hence $y \in Gx$, so $Gx = X$. Hence the first condition in iii. implies the third one and therefore also the second one. Vice versa, assume the second condition, and take $x \in X$ with $Gx = X$. Given any $y, z \in X$, this means $g, h \in G$ exist with $y = gx$ and $z = hx$. Then $x = (h^{-1}h)x = h^{-1}z$, and therefore $y = g(h^{-1}z) = (gh^-1)z$, which shows that the action is transitive.

So indeed the three conditions are equivalent.
iv.: We show the implication $\Rightarrow$ in the first equivalence: since $y \in Gy$, the condition $Gx = Gy$ implies $y \in Gx$.
Next we prove $\Leftarrow$ in the first equivalence. The condition $y \in Gx$ means that one can take $g \in G$ with $y = gx$, so also $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = x$. This will be used to show $Gx = Gy$. Indeed, let $z \in Gx$. Then we have $h \in G$ such that $z = hx$, and hence $z = h(g^{-1}y) = (hg^{-1})y \in Gy$. This shows $Gx \subseteq Gy$. Similarly, let $w \in Gy$. Then $w = h_1y$ for some $h_1 \in G$, and $w = h_1(gx) = (h_1g)x \in Gx$. So $Gy \subseteq Gx$ and hence $Gx = Gy$.
It remains to show the second equivalence in iv.
$\Rightarrow$: since $y \in Gy$, if Gx and Gy have no intersection then certainly $y \notin Gx$.
$\Leftarrow$: suppose $z \in Gx \cap Gy$. This means $g, h \in G$ exist with $gx = z = hy$. Then $y = (h^{-1}g)x \in Gx$, contradicting the assumption.
v.: $h \in G_{gx} \Leftrightarrow h(gx) = gx \Leftrightarrow (g^{-1}hg)x = x \Leftrightarrow g^{-1}hg \in G_x \Leftrightarrow h \in gG_xg^{-1}$. ∎

VI.3.8 Corollary. *Any G-set X is a disjoint union of orbits: $X = \cup Gx$.*

Proof. Any $x \in X$ is in an orbit, namely $x \in Gx$. If two orbits Gx, Gy are not disjoint, the proof of Theorem VI.3.7 iv shows $y \in Gx$ and therefore by the same theorem $Gx = Gy$. ∎

Recall (Definition VI.2.1) that the set of left cosets of a subgroup H in a group G is denoted G/H. We will use this in the case of a G-set X and, for $x \in X$, the subgroup $G_x \subseteq G$.

VI.3.9 Theorem. *Suppose G is a group and X is a G-set. Let $x \in X$. Then*

$$gG_x \mapsto gx \quad G/G_x \longrightarrow Gx$$

is a well-defined bijective map.

Proof. If $g, h \in G$ then $gx = hx \Leftrightarrow (h^{-1}g)x = x \Leftrightarrow h^{-1}g \in G_x \Leftrightarrow g \in hG_x \Leftrightarrow gG_x = hG_x$. This implies that the given map is well-defined. It remains to show that it is both surjective and injective.
Surjective: if $y \in Gx$, then $y = gx$ for some $g \in G$ hence y is the image of the left coset gG_x.
Injective: If gG_x and hG_x have the same image (which then by definition equals $gx = hx$, then the equivalences above show $gG_x = hG_x$. So the map is injective. ∎

In case the orbit Gx of an element x in the G-set X is finite, Theorem VI.3.9 implies in particular that the index $[G : G_x]$ of the stabilizer subgroup G_x in G is finite as well. More precisely:

VI.3.10 Corollary. *For any G-set X and any $x \in X$ one has $\#Gx = [G : G_x]$.*

Proof. VI.3.9 shows $\#Gx = \#(G/G_x)$ and by definition $\#(G/G_x) = [G : G_x]$. ∎

VI.3.11 Example. We will now show that many of the notions and results from Section VI.1 (on conjugation) can be interpreted as a special case of a group action.

Let G be a group. Then conjugation defines an action of G on itself. In other words, for $g, h, x \in G$ we have $exe^{-1} = x$ and $(gh)x(gh)^{-1} = g(hxh^{-1})g^{-1}$. The orbit of $x \in G$ consists of all elements gxg^{-1}, so it equals the conjugacy class C_x. Hence Corollary VI.3.8 implies Corollary VI.1.9.

The homomorphism $f : G \to S_X$ that comes with any group action, is in the present case given by $g \mapsto \gamma_g$, and Theorem VI.1.2 implies that indeed this is a homomorphism.

Is $x \in G$, then its stabilizer consists of all $g \in G$ satisfying $gxg^{-1} = x$. Hence the stabilizer equals the subgroup $N(x)$ described in Theorem VI.1.13. Moreover, the formula in VI.1.13 states that in case G is finite, its order is given in terms of the size of an orbit and a stabilizer. Hence this is a special case of Corollary VI.3.10 (using Theorem VI.2.3).

The fixpoints of the conjugation action of G on itself are by definition the $x \in G$ satisfying $gxg^{-1} = x$ for all $g \in G$. Multiplying on the right by g, this condition reads $gx = xg$ for all $g \in G$. In other words, the set of fixpoints equals the center $\mathcal{Z}(G)$ of G (see Exercise 7 on page 32). Note that the kernel of the homomorphism $f : G \to S_G$ also equals the center:

$$\gamma_g = f(g) = \mathrm{id}_G \Longleftrightarrow gxg^{-1} = x \text{ for all } x \in G \Longleftrightarrow g \in \mathcal{Z}(G).$$

∎

If both the group G and the G-set X are finite, there is a nice and useful formula for the *number* of orbits in X. Usually this is called Burnside's formula or Burnside's lemma. The English mathematician William Burnside (1852–1927) in 1897 published a textbook *Theory of Groups of Finite Order* in which he stated the result and attributes it to Frobenius. Indeed, the German mathematician Ferdinand Georg Frobenius (1849–1917) proved the formula 10 years earlier. In his paper however, many footnotes show that he is perfectly aware that the French mathematician Augustin-Louis Cauchy (1789–1857) already in 1845 mentions the result in the *Comptes Rendus* of the French Academy of Sciences.

To state the formula, one more concept is needed.

VI.3.12 Definition. Given a group G and a finite G-set X, the *permutation character* of the action is the function $\chi : G \to \mathbb{Z}$ given by

$$\chi(g) := \#\{x \in X \,:\, gx = x\}$$

In other words, the function χ assigns to any $g \in G$ the number of fixpoints in X of g. Exercise 7 on page 70 hints at some background of this function.

VI.3.13 Theorem (The orbit-counting formula)**.** *Let G be a finite group acting on a finite G-set X. The number of orbits in X under G is given by*

$$\#\text{orbits} = \frac{1}{\#G} \sum_{g \in G} \chi(g).$$

Proof. First we write the number of orbits as a sum over the elements of X. Let $x \in X$. Then for each $y \in Gx$ we have $Gx = Gy$, hence we obtain

$$\sum_{y \in Gx} \frac{1}{\#Gy} = 1.$$

Since X is a disjoint union of orbits (Corollary VI.3.8), it follows that

$$\#\text{orbits} = \sum_{x \in X} \frac{1}{\#Gx} = \sum_{x \in X} \frac{1}{[G : G_x]} = \frac{1}{\#G} \sum_{x \in X} \#G_x,$$

where also Corollary VI.3.10 was used. The proof will now be finished by writing $\#G_x$ as a sum over the elements of G and interchanging the summation over X and over G in the resulting formula for #orbits. To this end, for $g \in G$ and $x \in X$ put

$$\delta_{g,x} := \begin{cases} 1 \text{ if } gx = x; \\ 0 \text{ otherwise.} \end{cases}$$

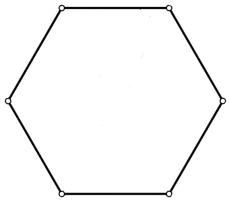
Observe for x fixed that $\sum_{g \in G} \delta_{g,x} = \#G_x$ and for g fixed $\sum_{x \in X} \delta_{g,x} = \chi(g)$ (the number of fixpoints of g). As a consequence

$$\#\text{orbits} = \frac{1}{\#G} \sum_{x \in X} \#G_x = \frac{1}{\#G} \sum_{x \in X} \sum_{g \in G} \delta_{g,x} = \frac{1}{\#G} \sum_{g \in G} \sum_{x \in X} \delta_{g,x} = \frac{1}{\#G} \sum_{g \in G} \chi(g),$$

finishing the proof. ∎

The orbit-counting formula tells us that to find the number of orbits, one determined for each element of the group its number of fixpoints and then one takes the average of the $\#G$ numbers written down.

VI.3.14 Example. Suppose we have beads of three different colors and we wish to make a necklace consisting of six beads. How many 'different' configurations of beads are possible?



To make the formulation of this problem more precise, one takes a regular hexagon and demands that the beads/colors should be placed on the vertices of it. Two configurations will be called 'the same' if one can be obtained from the other by simply rotating or reflecting the hexagon. In other words, if the symmetry group of the hexagon (which is the group D_6) maps the one configuration to the other.

The notion 'group action' allows one to state the problem as follows. Let X be the set of all possible configurations. Then $\#X = 3^6 = 729$, because each of the 6 vertices can be given three possible colors. The group D_6 acts on X: any configuration, under a rotation or reflection in D_6, results in another (possibly the same) one, and indeed this satisfies the requirement of being a group action. 'The same' configurations are precisely the ones in the same orbit under this action, so our problem is to determine the number of orbits. We do this by using Theorem VI.3.13, so by computing the average number of fixpoints. In our case this means: average number of fixed configurations.

Clearly $\mathrm{id} \in D_6$ fixes all of the 729 configurations. Next, consider a reflection in a line through the midpoints of two opposite edges. To be fixed by this reflection, the three vertices on one side of this line should have the same color as their reflections. Since one can color the first three vertices arbitrarily, this leads to $3^3 = 27$ fixed configurations. Note that there are 3 reflections of this type.

Now take a reflection in a line passing through two opposite vertices. To have a configuration fixed by this, the two vertices on the line can be colored arbitrarily, and the two on one side of the line again need to have the same color as their reflections. So this results in $3^4 = 81$ fixed configurations (and there are 3 reflections of this kind).

It remains to consider the rotations. The ones over ± 60 degrees send any vertex to a neighbour, hence for a configuration to be fixed, all vertices must have the same color. There are 3 such configurations. Similarly, to be fixed under a rotation over ± 120 degrees, the first, third, and fifth vertex need to have the same color, and so do the second, fourth, and sixth. Hence this yields $3^2 = 9$ configurations. Finally there is the rotation over 180 degrees. In this case opposite vertices need to have the same color, and $3^3 = 27$ configurations have this property.

In total, this gives

$$\#\text{different configurations} = \frac{729 + 3\cdot 27 + 3\cdot 81 + 2\cdot 3 + 2\cdot 9 + 27}{12} = 92.$$

VI.4 Sylow theory

Recall that by Lagrange's theorem, the order of a subgroup of a finite group G is a divisor of the order of G. However, the converse does not hold. For instance, the group A_4 has 12 elements, but no subgroup of order 6, as is readily verified. We will see that, nevertheless, there is a partial converse to Lagrange's theorem if we restrict to prime power divisors of the order of G.

VI.4.1 Definition. (After P.L.M. Sylow, Norwegian mathematician, 1832–1918.) Let G be a finite group and let p be a prime dividing the order of G. Write

$\#G = p^n \cdot m$, where $n \geq 1$ and $\gcd(p,m) = 1$. A *Sylow p-group* in G is a subgroup $H \subset G$ with $\#H = p^n$. We define $n_p(G)$ to be the number of pairwise distinct Sylow p-groups in G.

VI.4.2 Example. Take a prime p and $n,m \geq 1$ with $\gcd(p,m) = 1$. Then the group $G = \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ consists of $p^n m$ elements. There exists exactly one Sylow p-group in G, namely $H := \mathbb{Z}/p^n\mathbb{Z} \times \{\overline{0}\}$. Indeed, H is clearly a Sylow p-group in G. Suppose that $H' \leqslant G$ is also such a group, and let $h \in H'$. Then h is of the form $h = (a \bmod p^n, b \bmod m) \in H'$, and its order $\operatorname{ord}(h)$ divides $\#H' = p^n$. Moreover $\operatorname{ord}(b \bmod m) \mid \operatorname{ord}(h) \mid p^n$, and also $\operatorname{ord}(b \bmod m)$ divides $\#\mathbb{Z}/m\mathbb{Z} = m$. So $\operatorname{ord}(b \bmod m) \mid \gcd(p^n, m) = 1$, implying $b \bmod m = \overline{0}$. This shows $H' \subset H$. Since $\#H' = p^n = \#H$ we conclude $H' = H$. ∎

The following powerful theorem has several important applications. For instance, one can deduce statements about possible groups of a given order. However, the proof is rather involved.

VI.4.3 Theorem. *Let G be a finite group and let p be a prime dividing the order of G. Write $\#G = p^n \cdot m$, where $n \geq 1$ and $\gcd(p,m) = 1$.*

1. *The group G contains a Sylow p-group.*
2. *We have $n_p(G) \equiv 1 \bmod p$.*
3. *If H and H' are Sylow p-groups in G then $H' = \gamma_a(H)$ for some $a \in G$.*
4. *We have $n_p(G) \mid m$.*

Proof. Clearly 1. claims that $n_p(G) \neq 0$; hence 1. follows if we show the claim in 2. To this end, we study the collection of all subsets of G consisting of p^n elements.

If $H \leqslant G$ is a Sylow p-group and $g \in G$ then Hg is one of the sets under consideration. Theorem VI.2.3 shows that for given H there are $\#G/\#H = m$ such sets Hg. If $x \in G$, then we have $xHg = Hg$ precisely when $xH = H$; and this holds if and only if $x \in H$. So if we set $V := Hg$, then we retrieve H as the set of all $x \in G$ with $xV = V$.

Now suppose $V \subset G$ is a subset with p^n elements, and moreover suppose that the subgroup $G_V := \{x \in G \mid xV = V\} \leqslant G$ also contains p^n elements. If $v \in V$ and $x \in H := G_V$, then $xV = V$, hence $xv \in V$. So $Hv \subset V$, and because $\#Hv = p^n = \#V$ we have $V = Hv$. We conclude that every set V with the given properties has the form Hg, with $g \in G$ and H a Sylow p-group. In total there are $n_p(G) \cdot m$ such sets.

Now consider a subset $V \subset G$ such that $\#V = p^n$, but $\#G_V \neq p^n$. Just as in the proof of Theorem III.2.8, $V \subset G$ is a disjoint union of subsets $G_V \cdot v$, each having $\#G_V \neq p^n$ elements, and $\#G_V \mid \#V = p^n$. So $\#G_V = p^k$ for some $k < n$. Writing $\mathscr{P}$ for the collection of all such sets V, we have

$$\binom{p^n m}{p^n} = n_p(G)m + \#\mathscr{P}.$$

We claim that $\#\mathscr{P} \equiv 0 \bmod p$. Indeed, let $V \in \mathscr{P}$. Then, if $x \in G$, we have $xV \in \mathscr{P}$ as well, since $\#xV = \#V = p^n$, and $g \cdot (xV) = xV$ precisely when $x^{-1}gxV = V$. So $G_{(xV)} = \gamma_{x^{-1}}(G_V)$, which implies that $\#G_V = \#G_{(xV)}$. Now $x,y \in G$ satisfy

$$xV = yV \Leftrightarrow y^{-1}xV = V \Leftrightarrow y^{-1}x \in G_V \Leftrightarrow xG_V = yG_V.$$

Therefore precisely $[G : G_V]$ pairwise different sets $xV \in \mathscr{P}$ exist for a given $V \in \mathscr{P}$. But $V \in \mathscr{P}$ implies that $[G : G_V] = p^n m/p^k \equiv 0 \bmod p$. In this way $\mathscr{P}$ is partitioned into sub-collections, each containing ℓp sets V, where $\ell \in \mathbb{Z}$. This shows that $\#\mathscr{P} \equiv 0 \bmod p$.

We conclude

$$\binom{p^n m}{p^n} = n_p(G)m + \#\mathscr{P} \equiv n_p(G)m \bmod p.$$

Since $\gcd(m,p) = 1$, we know that $\overline{m} = m \bmod p$ is a unit in $\mathbb{Z}/p\mathbb{Z}$, and therefore

$$n_p(G) \bmod p = \overline{m}^{-1} \cdot \binom{p^n m}{p^n} \bmod p.$$

This shows that $n_p(G) \bmod p$ depends only on p, n, and m, and *not* on the actual group G. In particular, choosing $G = \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ one finds using Example VI.4.2 that $n_p(G) \bmod p = 1 \bmod p$. This proves 2. and hence also 1.

We now show 3. Let H, H' be Sylow p-groups in G. By definition, there are $[G : H] = m$ distinct subsets gH in G, and G is their union. We partition this collection of sets gH into two classes $\mathscr{H}_1, \mathscr{H}_2$ as follows: $gH \in \mathscr{H}_1$ if $hgH = gH$ for all $h \in H'$, and $gH \in \mathscr{H}_2$ otherwise. By construction $m = \#\mathscr{H}_1 + \#\mathscr{H}_2$. We show, in fact in a way quite similar to what was done for the collection $\mathscr{P}$ above, that $\#\mathscr{H}_2 \equiv 0 \bmod p$. Let $gH \in \mathscr{H}_2$. Then $H'' := \{h \in H' \mid hgH = gH\}$ is a subgroup of H'. The definition of $\mathscr{H}_2$ implies $H'' \neq H'$, so $p \mid [H' : H'']$. Given $gH \in \mathscr{H}_2$ and $h \in H'$, consider hgH. This is clearly an element of $\mathscr{H}_2$. Moreover, for $h_1, h_2 \in H'$ one finds

$$h_1 gH = h_2 gH \Leftrightarrow h_2^{-1} h_1 gH = gH \Leftrightarrow h_2^{-1} h_1 \in H'' \Leftrightarrow h_1 H'' = h_2 H''.$$

So there are $[H' : H'']$ distinct sets hgH if we let h run through H'. This partitions $\mathscr{H}_2$ into disjoint subsets, each having ℓp elements, where $\ell \in \mathbb{Z}$. So $\#\mathscr{H}_2 \equiv 0 \bmod p$. This implies

$$\#\mathscr{H}_1 \equiv m \bmod p \neq 0 \bmod p,$$

hence in particular $\mathscr{H}_1$ is nonempty. So there exists a set gH such that $hgH = gH$ for all $h \in H'$. In other words: $g^{-1}hg \in H$ for all $h \in H'$, which means $H' \subset \gamma_g(H)$. This proves 3.

Finally we prove 4. Take H a Sylow p-group in G. There are N such groups, and we have already shown that each of them can be written as $\gamma_g(H)$ for some $g \in G$. Define

$$N(H) := \{g \in G \mid \gamma_g(H) = H\}.$$

This is a subgroup of G, and writing $G = \cup g_i N(H)$ one finds that the Sylow p-groups in G are exactly the groups $\gamma_{g_i}(H)$, and these are pairwise distinct. You should check the details of this argument yourself. So $n_p(G) = [G : N(H)]$. Since $H \leqslant N(H)$ we have $\#N(H) = [N(H) : H] \cdot \#H$ and

$$n_p(G) = [G : N(H)] = \#G/\#N(H) = \#G/([N(H) : H] \cdot \#H) \mid \#G/\#H = m.$$

This finishes the proof. ∎

VI.4.4 Remark. In fact, this proof may be phrased more conveniently using the terminology of 'group actions' (Section VI.3): the group G acts on the set X consisting of all subsets $V \subset G$ with $\#V = p^n$, by multiplication: $(g, V) \mapsto gV$. The proof shows that $V \in X$ exist with stabilizer G_V of order p^n, hence it exhibits p-Sylow groups as stabilizers for this action.

VI.4.5 Corollary. *For p prime and $n, m > 0$ with $\gcd(p,m) = 1$ we have*

$$\binom{p^n m}{p^n} \equiv m \bmod p.$$

Proof. During the proof of Theorem VI.4.3 we showed that $\binom{p^n m}{p^n} \equiv n_p(G)m \bmod p$ and $n_p(G) \equiv 1 \bmod p$. This implies the corollary. ∎

VI.4.6 Example. S_4 has $24 = 3 \cdot 8$ elements. By Theorem VI.4.3 the number of Sylow 3-groups in S_4 divides 8, and has the form $3k+1$. So their number is either 1 or 4. Each subset $\{(1),(a\ b\ c),(a\ c\ b)\}$ with $a \neq b, b \neq c, c \neq a$ is such a subgroup, and this yields 4 of them.

The number of Sylow 2-groups in S_4 is odd and it divides 3. So we have 1 or 3 of them. If H is such a group then $\#H = 8$, so every element in H has an order dividing 8. Therefore only 4-cycles, 2-cycles, products of two disjoint 2-cycles, and the identity can possibly belong to H. There can be at most two 2-cycles in H, and if there are then they are disjoint, since otherwise H would contain a product $(a\ b)(b\ c) = (a\ b\ c)$, which is impossible. It is also not possible that H contain only one 2-cycle. Indeed, in that case every subgroup $\sigma H \sigma^{-1}$ also contains only one 2-cycle, and since all 2-cycles are conjugate this means there are at least as many Sylow 2-groups in S_4 as there are 2-cycles. Since there are six 2-cycles and at most three Sylow 2-groups, this is impossible. So H contains either no, or exactly two (disjoint) 2-cycles.

The number of 4-cycles in H is even, because a 4-cycle differs from its inverse and either both or none of the pair is in H. The square of a 4-cycle is a product of two disjoint 2-cycles, and a 4-cycle and its inverse have the same square and all other 4-cycles do not have this same square. As a consequence H contains exactly one 4-cycle and its inverse. Hence H contains at least one 4-cycle and its inverse, because otherwise H would contain at most 6 elements. It cannot contain more than one 4-cycle and its inverse because otherwise one can show by conjugating one of them by powers of the other one that *all* 4-cycles are in H, and then all products of two disjoint 2-cycles as well. In that case $\#H \geq 1+6+3 = 10$.

We conclude that H consists of the identity, a 4-cycle and its inverse, all three products of two disjoint 2-cycles, and two 2-cycles. A small computation yields three such (conjugate) groups. One of them is given by

$$\{(1),(1\,2\,3\,4),(1\,4\,3\,2),(1\,2)(3\,4),(1\,3)(2\,4),(1\,4)(2\,3),(1\,3),(2\,4)\}.$$

Here is a much less elaborate way to construct this example: the symmetry group D_4 of the square has 8 elements. These elements permute the vertices of the square. So D_4 can be considered as a subgroup of S_4. Explicitly: in the x,y-plane take the square with vertices $(\pm 1, \pm 1)$. The vertex in the i-th quadrant is denoted i. Rotating counter clockwise by 90 degrees yields the permutation (1 2 3 4) on the vertices. Reflection in the x-axis corresponds to (1 4)(2 3). Reflection in the diagonal $x+y=0$ yields (1 3), et cetera. ▬■

The next two results illustrate the possible use of Theorem VI.4.3.

VI.4.7 Theorem. *Suppose $p \neq q$ are primes with $p \not\equiv 1 \bmod q$ and $q \not\equiv 1 \bmod p$, and G is a group with $\#G = pq$. Then $G \cong \mathbb{Z}/pq\mathbb{Z}$.*

Proof. By Theorem III.2.11 any $g \in G$ satisfies $\operatorname{ord}(g) \in \{1, p, q, pq\}$. The only element of order 1 is $e \in G$. If $\operatorname{ord}(g) = p$, then $\langle g \rangle$ is a Sylow p-group in G. The number of such groups divides q, so it is 1 or q. Moreover, the number is $\equiv 1 \bmod p$. Since $q \not\equiv 1 \bmod p$, the number of Sylow p-groups in G equals 1. Every element of G of order p is in this Sylow p-group, so there are at most $p-1$ such elements. (In fact there are precisely $p-1$ of them, but we will not need this.)

The same reasoning shows that at most $q-1$ elements in G have order q. Since $1+p-1+q-1 < pq = \#G$, the group G must contain elements of order pq. If $g \in G$ is any such element, then $\langle g \rangle = G$, and $g \mapsto 1 \bmod pq$ yields an isomorphism $G \cong \mathbb{Z}/pq\mathbb{Z}$ (check the details yourself). ■

VI.4.8 Example. Applying Theorem VI.4.7 with $p = 3$ and $q = 5$, it follows that up to isomorphism only one group exists with 15 elements, namely $\mathbb{Z}/15\mathbb{Z}$. ▬■

VI.4.9 Theorem. (Augustin-Louis Cauchy, French mathematician, 1789–1857)
If G is a finite group and if p is a prime dividing the order of G, then there exists $g \in G$ with $\operatorname{ord}(g) = p$.

Proof. Take a Sylow p-group $H \leqslant G$. Such a group exists because of Theorem VI.4.3, and $\#H = p^n$ with $n \geq 1$. Take $x \in H$ such that $x \neq e$. Then $\operatorname{ord}(x) \neq 1$, and $\operatorname{ord}(x) \mid p^n$. Hence $\operatorname{ord}(x) = p^\ell$ with $1 \leq \ell \leq n$. Then $g := x^{p^{\ell-1}}$ has $\operatorname{ord}(g) = p$, as required. ∎

VI.4.10 Remark. A proof of Theorem VI.4.9 which does not use Sylow theory is sketched in Exercise 12 below.

VI.4.11 Example. The requirement in Theorem VI.4.9 that p divides the number of elements in the group G is necessary because of Theorem III.2.11. The requirement that p is prime is necessary as well. For example, the group D_4 has order 8, yet no element of order 8 exists in this group. More generally, if a group G consists of n elements, then an element of order n exists if and only if $G \cong \mathbb{Z}/n\mathbb{Z}$. In particular, this implies that G is abelian. So in a non-abelian group with n elements, no element of order n exists.

As another example, S_4 consists of 24 elements. The positive divisors of 24 are $\{1,2,3,4,6,8,12,24\}$. The divisors that occur as the order of some element in S_4, are $\{1,2,3,4\}$. ▬∎

VI.5 Exercises

1. Determine the number of elements of all conjugacy classes in S_6.
2. Find the conjugacy classes in A_6, and determine for each of them the number of elements.
3. Prove the second assertion in Theorem VI.1.14.
4. In the group D_n we have ρ = 'rotate counter clockwise by $2\pi/n$', and σ = 'reflect in the x-axis'.
   (a) Show that $\sigma\rho\sigma = \rho^{-1}$.
   (b) Show that every $\tau \in D_n$ can be written as $\rho^a\sigma^b$, with $0 \le a < n$ and $0 \le b \le 1$.
   (c) Take n odd. Find a conjugacy class in D_n consisting of n elements, another one containing 1 element, and show there are $(n-1)/2$ remaining classes C_τ containing 2 elements each.
   (d) Now take n even. Show that D_n has two conjugacy classes containing 1 element, $(n-2)/2$ conjugacy classes containing 2 elements, and two containing $n/2$ elements.
5. Suppose that G is a finite group with $\#G = n$, and G contains precisely 3 conjugacy classes.
   (a) Show that $n = 1 + a + b$, with $1 \le a \le b$ and $a|n$ and $b|n$.
   (b) Find all solutions to the equation in (a). (E.g., divide by n, and verify that $b \le 3$ holds.)
   (c) Use Exercise 13 that any non-commutative group with 6 elements is isomorphic to S_3, and show that $G \cong \mathbb{Z}/3\mathbb{Z}$ or $G \cong S_3$. Check that these groups indeed have precisely 3 conjugacy classes.
6. Given a group G, a subgroup $H \subset G$ and any $g \in G$, show $[G:H] = [G:\gamma_g(H)]$.
7. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set consisting of n elements. Using X one defines a complex vector space V_X as follows. The elements of V_X are the functions $\alpha: X \to \mathbb{C}$. Addition and scalar multiplication are done pointwise, so if $\alpha, \beta \in V_X$, then $\alpha + \beta$ is the function that maps x_j to $\alpha(x_j) + \beta(x_j)$, and for $\lambda \in C$ the function $\lambda\alpha$ maps x_j to $\lambda\alpha(x_j)$.
   Suppose now X is a G-set, for some group G. For $g \in G$ define $\rho(g): V_X \to V_X$ as follows. Given $\alpha \in V_X$, let $\rho(g)(\alpha) \in V_X$ be the map that sends $x_j \in X$ to $\alpha(gx_j)$.
   (a) Show that $\rho(g)$ is a linear map.
   (b) Show that $\rho(g)$ is invertible.
   (c) Show that $\rho: G \to \mathrm{GL}(V_X)$ is a homomorphism.
   (d) Show that the trace of the linear map $\rho(g)$ equals the integer $\chi(g)$ given in Definition VI.3.12.
   (e) Conclude from the above that χ is a 'class function', which means that $\chi(g) = \chi(hgh^{-1})$ (so χ is constant on conjugacy classes).
8. In how many 'different' ways can the vertices of a regular 10-gon be colored, using the four colors RGYB (red, green, yellow, blue) if we demand that some color is used 4 times, another one 3 times, a third one twice, and then clearly the remaining one only once? Here 'different' means that it is not possible to change one configuration into the other using an element of D_{10}.
9. Show that a finite *abelian* group G contains a unique Sylow p-group for every prime p with $p|\#G$.
10. For every prime p, find the number of Sylow p-groups in S_5.
11. This exercise describes the Sylow p-groups in S_6.
    (a) Show that Sylow p-groups in S_6 do not exist for $p > 5$.

(b) Show that the Sylow 2-groups in S_6 are isomorphic to $D_4 \times \mathbb{Z}/2\mathbb{Z}$. Show there are $\binom{6}{2} \cdot 3 = 45$ such groups. (Consider a Sylow 2-group in S_4, and $(5\,6) \in S_6$.)

(c) Show that the Sylow 3-groups are isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, and that there are $\binom{6}{3}/2 = 10$ of them. (Use disjoint 3-cycles.)

(d) Show that there are 36 Sylow 5-groups, isomorphic to $\mathbb{Z}/5\mathbb{Z}$.

12. This exercise provides an alternative proof of Theorem VI.4.9. The argument is due to the British/Canadian mathematician John McKay.
Let G be a finite group and let p be a prime with $p|n = \#G$. Consider

$$\mathscr{D} = \{(a_1, \ldots, a_p) \in G \times G \times \ldots \times G \mid a_1 a_2 \ldots a_p = e\}$$

and its subsets $\mathscr{D}_1 = \{(a_1, \ldots, a_p) \in \mathscr{D} \mid a_1 = a_2 = \ldots = a_p\}$ and $\mathscr{D}_2 = \mathscr{D} \setminus \mathscr{D}_1$.

(a) Show that $\#\mathscr{D} = n^{p-1} \equiv 0 \bmod p$.

(b) Show that if $(a_1, a_2, \ldots, a_p) \in \mathscr{D}$, then also $(a_2, \ldots, a_p, a_1)$ and $(a_p, a_1, a_2, \ldots)$ are in $\mathscr{D}$ and, more generally, so are all elements obtained by cyclically permuting the entries in $(a_1, a_2, \ldots, a_p)$.

(c) Show that if $(a_1, a_2, \ldots, a_p) \in \mathscr{D}_2$, then the p elements of $\mathscr{D}_2$ obtained by cyclically permuting the entries, are pairwise distinct.

(d) Prove that $\#\mathscr{D}_2 \equiv 0 \bmod p$.

(e) Prove that $\#\mathscr{D}_1 \geq p$, and conclude that the number of elements in G of order p is congruent to $p-1$ modulo p. In particular, such elements exist.

(f) For which of the steps in the proof is it crucial that p is prime?

13. Let G be a group with $\#G = 6$.

(a) Explain that $a, b \in G$ exist with $\mathrm{ord}(a) = 2$ and $\mathrm{ord}(b) = 3$.

(b) Show that if a, b as in (a) satisfy $\gamma_b(a) = a$, then $\mathrm{ord}(ab) = 6$ and $G \cong \mathbb{Z}/6\mathbb{Z}$.

(c) Show that if a, b as in (a) satisfy $\gamma_b(a) \neq a$, then C_a consists of 3 elements all having order 2, and $G \cong S_3$.

VII NORMAL SUBGROUPS AND FACTOR GROUPS

Reviewing Chapter II from a group theoretic perspective, we constructed a group $\mathbb{Z}/N\mathbb{Z}$ starting from $\mathbb{Z}$ and its subgroup $N\mathbb{Z}$. The *elements* of this new group are the *residue classes* $a+N\mathbb{Z}$. Theorem II.1.6 and the succeeding definition and remark show that the group law on $\mathbb{Z}$ (addition) gives rise to a group law (addition as well) on these residue classes. This chapter discusses an extension of this construction for arbitrary groups G, using a subgroup $H \leqslant G$. So in particular we study how one may set up calculations with the classes gH for $g \in G$. As it turns out, we obtain a group structure on the collection of sets $\{gH \mid g \in G\}$, similar to the case $G = \mathbb{Z}$ and $H = n\mathbb{Z}$, if and only if H satisfies a certain condition. In particular, we will see that this always works if G is abelian.

We encourage the reader to review the material from Section II.1 before studying the present chapter, as several constructions and results discussed below were already treated there in a concrete example.

VII.1 Normal subgroups

Given a group G and a subgroup H, Theorem VI.1.2 says that for $a \in G$ the conjugate $\gamma_a(H) = aHa^{-1}$ is also a subgroup of G. Moreover H and $\gamma_a(H)$ are isomorphic, but in general $H \neq \gamma_a(H)$. For example $H := \{(1),(1\,2)\}$ is a subgroup of S_3. For $a = (1\,3)$ we find $\gamma_a(H) = \{(1),(2\,3)\} \neq H$.

VII.1.1 Definition. A subgroup H of a group G is called *normal* if $H = aHa^{-1}$ for all $a \in G$.

VII.1.2 Example. In a commutative group G every subgroup H is normal, since in this case $aha^{-1} = aa^{-1}h = h$ for all $a \in G$ and $h \in H$, so $aHa^{-1} = H$. ∎

VII.1.3 Example. In the dihedral group D_n the rotations form a subgroup. Regarded as linear maps on $\mathbb{R}^2$ the rotations in D_n are exactly the elements of D_n with determinant 1. If ρ is a rotation and $a \in D_n$, then

$$\det(a\rho a^{-1}) = \det(a)\det(\rho)\det(a^{-1}) = \det(a)\det(a)^{-1} = 1,$$

so $a\rho a^{-1}$ is also a rotation. Hence the rotations form a normal subgroup. ∎

VII.1.4 Example. We determine all normal subgroups in S_4. Take a normal $H \leqslant S_4$ and let $\sigma \in H$. Since $H = \tau H \tau^{-1}$ for every $\tau \in S_n$, H contains the conjugacy class C_σ of σ. Using that S_4 is a disjoint union of conjugacy classes (Theorem VI.1.8),

also H is a disjoint union of conjugacy classes in S_4. The conjugacy classes in S_4 have $1, 6, 8$, and 3 elements, respectively. Obviously $(1) \in H$, so $\#H$ is a sum of some of the integers in $\{1, 3, 6, 8\}$ having 1 as a summand. Moreover $\#H | \#S_4 = 24$ by Theorem III.2.8. This leaves us with only a few possibilities:

1. $\#H = 1$, so $H = \{(1)\}$. Indeed this is a normal subgroup.
2. $\#H = 1 + 3 = 4$, so $H = \{(1), (12)(34), (13)(24), (14)(23)\}$. This is indeed a normal subgroup in S_4 (check this yourself).
3. $\#H = 1 + 3 + 8 = 12$. In this case H consists of the identity, the products of two disjoint 2-cycles, and all 3-cycles. So $H = A_4$ which is a normal subgroup in S_4, as is readily verified.
4. $\#H = 1 + 3 + 8 + 12 = 24$, so $H = S_4$.

We see that, although S_4 has many subgroups, it only has two proper normal subgroups. ▬

VII.1.5 Example. We have already mentioned that A_4 is normal in S_4. More generally A_n is normal in S_n for all n. This follows from the fact that for permutations σ, τ one has $\epsilon(\sigma) = \epsilon(\tau\sigma\tau^{-1})$. (Alternatively, conjugating a product of disjoint cycles yields a product of disjoint cycles of the same type. In particular this does not affect the sign, hence $\tau A_n \tau^{-1} = A_n$.) ▬

VII.1.6 Example. Let G be a finite group and $\#G = p^n m$ with p prime, $n \geq 1$, and $\gcd(p, m) = 1$. Consider a Sylow p-group $H \leqslant G$. By Theorem VI.4.3 all conjugate groups aHa^{-1} for $a \in G$ are Sylow p-groups as well, and we obtain Sylow p-groups in G in this way. We conclude that H is normal in G if and only if there is only one Sylow p-group in G. In many instances this condition may be verified using the two divisibility properties for the number of Sylow p-groups, as given in Theorem VI.4.3. ▬

The following useful lemma has in fact already been used and derived in a number of earlier situations.

VII.1.7 Lemma. *If H is a subgroup of a group G and if $a, b \in G$, then $aH = bH$ if and only if $b^{-1}a \in H$.*

Proof. In the proof of Theorem III.2.8 it was shown that two sets aH, bH are either equal or disjoint. Now $e \in H$, hence $a = ae \in aH$, so $aH = bH$ is equivalent to $a \in bH$. In other words: $aH = bH$ if and only if $a = bh$ for some $h \in H$, which in turn is equivalent to $b^{-1}a = h \in H$. ■

VII.1.8 Theorem. *Let G be a group and let $H \leqslant G$ be a subgroup. The following statements are equivalent:*

1. *H is normal in G.*
2. *Every $a \in G$ satisfies $aH = Ha$.*
3. *For all $a \in G$ we have $aHa^{-1} \subset H$.*
4. *For all $a, b, c, d \in G$ with $aH = cH$ and $bH = dH$ we also have $abH = cdH$.*

Proof. 1. implies 2.: If $a \in G$, then $aHa^{-1} = H$. Multiplying this equality on the right by a yields $aH = Ha$.
2. implies 3.: Take $a \in G$ and $h \in H$. The assumption $aH = Ha$ implies $ah = h_1 a$ for some $h_1 \in H$, so $aha^{-1} = h_1 \in H$ which is what we need.
3. implies 4.: By Lemma VII.1.7 it suffices to show that if $c^{-1}a, d^{-1}b \in H$, then also $(cd)^{-1}(ab) = d^{-1}c^{-1}ab \in H$. Write $c^{-1}a = h_1 \in H$. Assuming 3. yields that $h_2 := d^{-1}h_1 d \in H$. Then also $d^{-1}c^{-1}ab = d^{-1}h_1 dd^{-1}b = h_2 d^{-1}b \in H$, finishing the

argument.
4. implies 1.: Let $h \in H$ and $a \in G$. One has $hH = eH$, hence using the assumption 4. also $ha^{-1}H = ea^{-1}H = a^{-1}H$. Lemma VII.1.7 therefore implies $aha^{-1} \in H$. This shows $aHa^{-1} \subset H$. Applying the argument to $a^{-1} \in G$ we also have $a^{-1}Ha \subset H$, and therefore $h = a(a^{-1}ha)a^{-1} \in aHa^{-1}$, so $H \subset aHa^{-1}$. This shows $H = aHa^{-1}$, so H is normal in G.
Hence the four assertions are equivalent. ∎

The fact that the rotations in D_n and the even permutations in S_n are normal subgroups, is a special case of the following.

VII.1.9 Theorem. *If G is a group and if H is a subgroup of G with $[G : H] = 2$, then $H \leqslant G$ is normal.*

Proof. The condition $[G : H] = 2$ implies that there exists $a \in G$ such that G is the disjoint union of H and Ha, hence $Ha = G \setminus H$. Since $a \notin H$, also H and aH are disjoint subsets of G. Again using $[G : H] = 2$ it follows from Remark VI.2.2 that $aH = G \setminus H$, so $aH = Ha$. Every subset $bH, Hb \subset G$ either equals H (in case $b \in H$) or equals aH (in case $b \notin H$). So $bH = Hb$ for all $b \in G$, and Theorem VII.1.8 shows that H is normal in G. ∎

VII.2 Factor groups

Theorem VII.1.8 implies that if a subgroup $H \leqslant G$ is normal and if $a, b \in G$, then the product abH is independent of the representative a for aH and b for bH; i.e. if we have $aH = cH$ or $bH = dH$ for some elements $c, d \in G$, then we get $abH = cdH$. In other words, the rule $(aH)\cdot(bH) := abH$ is a well-defined operation on the collection of sets aH, $a \in G$. (And vice versa, if H is *not* normal, then in general the result *will* depend on the element in G used for describing the set aH, so the product is not well-defined.)

VII.2.1 Example. Take $G = S_3$ and $H = \{(1), (1\,2)\} \subset S_3$. Then H is a subgroup of G, but H is not normal in G. Put $a = (1\,3)$ and $b = (1\,2\,3)$. Then

$$aH = \{(1\,3)(1), (1\,3)(1\,2)\} = \{(1\,3), (1\,2\,3)\}$$

and

$$bH = \{(1\,2\,3)(1), (1\,2\,3)(1\,2)\} = \{(1\,2\,3), (1\,3)\}.$$

So $aH = bH$ (as is also clear from $b^{-1}a = (1\,2) \in H$ using Lemma VII.1.7). However $a^2H = H$ and $b^2H = (1\,3\,2)H = \{(1\,3\,2), (2\,3)\} \neq a^2H$, which shows that in this case a multiplication on the sets $\{gH\}$ as above is not possible. ∎

VII.2.2 Definition. Given a group G and a normal subgroup $H \leqslant G$, the *factor group* G modulo H, which we denote by G/H, is the group whose elements are the sets aH for $a \in G$. The unit element is $H = eH$, and the group law is defined by $(aH)\cdot(bH) := abH$.

VII.2.3 Remark. The given multiplication on G/H is well-defined because of Theorem VII.1.8, and this uses the fact that H is normal in G. Since G is a group, it follows easily that G/H forms a group as well. For example, the inverse $(aH)^{-1}$ of an element $aH \in G/H$ equals $a^{-1}H$. Indeed, $(aH)\cdot a^{-1}H = eH$, which by definition is the unit element in G/H.

VII.2.4 Remark. The definition of the index shows that the total number of pairwise distinct sets aH equals $[G:H]$. So $\#(G/H) = [G:H]$. If G is a finite group, then Theorem VI.2.3 shows $\#(G/H) = [G:H] = \#G/\#H$.

VII.2.5 Example. The subgroup $H := N\mathbb{Z}$ is normal in $G = \mathbb{Z}$. The factor group is the group $\mathbb{Z}/N\mathbb{Z}$. This example shows in particular that a factor group of an infinite group may be finite. ▬

VII.2.6 Example. Let $n \geq 2$ and consider the normal subgroup $H := A_n$ in $G = S_n$. Since A_n has index 2 in S_n, the factor group S_n/A_n consists of two elements. In particular, $S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$ because up to isomorphism there is only one group consisting of two elements. We conclude that a factor group of a non-abelian group may be commutative.

The two elements of S_n/A_n are by definition two subsets of S_n. One consists of all *even* permutations and the other of all *odd* permutations. The group law in S_n/A_n is described by the rule 'even times even is even' and 'odd times even is odd' and 'even times odd is odd' and 'odd times odd is even'. ▬

If G is an abelian group, and if $H \leqslant G$, then G/H is abelian as well. But the converse does not hold (see Example VII.2.1). We now present a criterion to check when the factor group is abelian.

VII.2.7 Theorem. *If H is a normal subgroup of a group G, then the factor group G/H is abelian if and only if the element $a^{-1}b^{-1}ab$ is in H for all $a, b \in G$.*

Proof. By definition G/H is abelian if and only if $(aH)\cdot(bH) = (bH)\cdot(aH)$ for all $a, b \in G$. The latter holds if and only if $abH = baH$ or all $a, b \in G$. By Lemma VII.1.7 this last condition is equivalent to $a^{-1}b^{-1}ab = (ba)^{-1}ab \in H$ for all $a, b \in G$. ■

VII.2.8 Example. Let $n \geq 3$. As we noted, S_n/A_n is an abelian group. Hence by Theorem VII.2.7, we get for all permutations σ, τ that the product $\sigma^{-1}\tau^{-1}\sigma\tau$ is even. This, of course, is also clear from the fact that the sign ϵ is a homomorphism from S_n to a commutative group ($\{\pm 1\}$). Since $(a\ b)^{-1}(a\ c)^{-1}(a\ b)(a\ c) = (a\ b\ c)$ for pairwise distinct a, b, c and since every element of A_n can be written as a product of 3-cycles, it follows that if $H \leqslant S_n$ is normal and moreover S_n/H is abelian, then $A_n \subset H$, so $H = A_n$ or $H = S_n$. ▬

VII.2.9 Theorem. *Let H be normal in a group G. The assignment*

$$\pi : G \longrightarrow G/H \ : \ g \mapsto gH$$

defines a surjective homomorphism from G to G/H with $\ker(\pi) = H$.

Proof. For $a, b \in G$ one has $\pi(ab) = abH = (aH)\cdot(bH) = \pi(a)\pi(b)$. So π is a homomorphism. Any element in G/H has the form aH for some $a \in G$. Since $\pi(a) = aH$, we deduce that π is surjective. Finally, $a \in G$ satisfies $a \in \ker(\pi)$ if and only if $aH = eH$, so by Lemma VII.1.7 this happens if and only if $a \in H$. Hence $\ker(\pi) = H$, which completes the proof. ■

VII.2.10 Remark. The homomorphism π given in Theorem VII.2.9 is usually called the *canonical* homomorphism to a factor group.

If $H \leqslant G$ is a normal subgroup,then Theorem VII.2.9 shows that H is the kernel of the canonical homomorphism from G to G/H. In fact this statement has a converse.

VII.2.11 Theorem. *A subgroup H of a group G is normal if and only if H is the kernel of some homomorphism from G to another group.*

Proof. One direction follows from Theorem VII.2.9; we leave the other direction as a good and not too difficult exercise for the reader. ∎

VII.3 Simple groups

VII.3.1 Definition. A group $G = (G, \cdot, e)$ is called *simple* if $\{e\}$ and G are the only normal subgroups in G.

VII.3.2 Remark. If G is a simple group, if G' is any group, and if $f : G \to G'$ a homomorphism, then either f is injective or f is the constant map sending every element of G to the unit element of G'. Indeed, the kernel of f is normal in G, hence $\ker(f) = \{e\}$ (implying that f is injective) or $\ker(f) = G$ (meaning that everything is mapped to the unit element). This property of simple groups indicates that 'being simple' is a strong property.

VII.3.3 Remark. If $H \leqslant G$ is a proper normal subgroup, then, in a certain sense, G 'decomposes' into H and G/H. This uses the theory of group extensions, which we will not cover in this introductory course. Since the simple groups are precisely the ones for which no nontrivial decomposition of this kind exists, they form the 'building blocks' for all groups in this sense.

VII.3.4 Example. We determine all nontrivial finite abelian simple groups G. If G is such a group, and if p is a prime dividing $\#G$, then Theorem VI.4.9 shows the existence of an element $a \in G$ with $\mathrm{ord}(a) = p$. The subgroup $\langle a \rangle$ is normal in G (any subgroup of an abelian group is normal) and $\neq \{e\}$. Since G is simple, we must therefore have $G = \langle a \rangle \cong \mathbb{Z}/p\mathbb{Z}$. On the other hand, $\mathbb{Z}/p\mathbb{Z}$ is indeed simple, because the order of any subgroup has is a divisor of p and p is prime. We conclude that, up to isomorphism, the groups $\mathbb{Z}/p\mathbb{Z}$ are the only nontrivial simple finite abelian groups. ∎

VII.3.5 Remark. One of the main results in the modern theory of finite groups is a complete list of all finite simple groups (up to isomorphims, as usual). The list consists of some infinite 'families of simple groups' (such as the $\mathbb{Z}/p\mathbb{Z}$'s for p prime), and 26 more groups not appearing in any of the families. These additional ones are called the 'sporadic groups'. This list together with various properties of the groups is described in the book by J. Conway et al., *Atlas of finite simple groups*. Oxford: Clarendon Press, 1985. A digital version containing similar information can be found at `http://brauer.maths.qmul.ac.uk/Atlas/v3/`. The proof that the list is complete involved an enormous amount of collaboration to which over a hundred mathematicians contributed. In particular the American mathematician Daniel Gorenstein (1923–1992) deserves credit for this.

The largest sporadic group goes by the intriguing name 'the Monster'. This group consists of

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

elements.

Other finite simple groups include for example the groups $\mathrm{PSL}_n(\mathbb{Z}/p\mathbb{Z})$, where $n \geq 2$ and p is prime such that $(n,p) \neq (2,2)$ and $(n,p) \neq (2,3)$. These are the factor

groups G/H, with G the group $\mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$ of all $n \times n$ matrices with coefficients in $\mathbb{Z}/p\mathbb{Z}$ and determinant $\overline{1}$, and H the subgroup consisting of all matrices $\overline{a}I$ with $\overline{a}^n = \overline{1}$. Proving that these groups are indeed simple is not part if the present course. We *will* however show that the groups A_n for $n \geq 5$ are simple.

VII.3.6 Example. We show that A_5 is simple. Let $H \leqslant A_5$ be normal. If $\sigma \in H$, then also $\tau\sigma\tau^{-1} \in H$ for all $\tau \in A_5$. Hence H contains the conjugacy class C_σ of σ in A_5. It follows that H is a union of such conjugacy classes. These classes are pairwise disjoint and they contain $1, 12, 12, 15$, and 20 elements, respectively (compare Example VI.1.12). So

$$\#H = 1 + 12a + 15b + 20c$$

with $a \in \{0,1,2\}$ and $b, c \in \{0,1\}$. Moreover $\#H | \#A_5 = 60$. It is not hard to show that these conditions imply $\#H = 1$ or $\#H = 60$. Hence indeed A_5 is simple. ∎

VII.3.7 Theorem. *A_n is a simple group for every $n \geq 5$.*

Proof. The idea of the proof below is to show that a normal subgroup $H \neq \{(1)\}$ in A_n contains a 3-cycle. Then we deduce that H contains the conjugacy class of this 3-cycle in A_n. By Theorem VI.1.14 this conjugacy class contains *all* 3-cycles. And therefore Theorem IV.4.4 implies that $H = A_n$, so A_n is simple.

Let $n \geq 5$ and let $H \neq \{(1)\}$ be a normal subgroup of A_5. Take $\sigma \neq (1)$ in H. Put $\sigma = \sigma_1\sigma_2 \ldots \sigma_r$, where the σ_i are disjoint ℓ_i-cycles and $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_r \geq 2$.

If $\ell_1 \geq 4$, then let $\sigma_1 = (a_1\ a_2\ \ldots\ a_{\ell_1})$ and set $\tau := (a_1\ a_2\ a_3) \in A_n$. Since $H \leqslant A_n$ is normal, also $\sigma' = \tau\sigma\tau^{-1} \in H$. We take a closer look at σ'. The numbers a_1, a_2, a_3 occur in σ_1 and not in $\sigma_2, \ldots, \sigma_r$. Hence $\tau\sigma_i\tau^{-1} = \sigma_i$ for $i \geq 2$ and $\tau\sigma_1\tau^{-1} = (\tau(a_1)\,\tau(a_2)\,\ldots\,\tau(a_{\ell_1})) = (a_2\,a_3\,a_1\,a_4\,\ldots\,a_{\ell_1})$. Therefore

$$\sigma' = \tau\sigma\tau^{-1} = (\tau\sigma_1\tau^{-1})(\tau\sigma_2\tau^{-1})\ldots(\tau\sigma_r\tau^{-1}) = (a_2\,a_3\,a_1\,a_4\,\ldots\,a_{\ell_1})\sigma_2\ldots\sigma_r.$$

Now $\sigma^{-1}\sigma' = (a_{\ell_1}\,\ldots\,a_1)(a_2\,a_3\,a_1\,a_4\,\ldots\,a_{\ell_1}) = (a_1\,a_3\,a_{\ell_1}) \in H$. So in this case ($\ell_1 \geq 4$) the group H contains a 3-cycle, and we are done.

If $\ell_1 = \ell_2 = 3$ then write $\sigma_1 = (a_1\ a_2\ a_3)$ and $\sigma_2 = (b_1\ b_2\ b_3)$. Conjugation by $\tau = (a_1\ a_2\ b_1) \in A_n$ yields $\sigma' = (a_2\ b_1\ a_3)(a_1\ b_2\ b_3)\sigma_3\ldots\sigma_r \in H$. Hence also $\sigma^{-1}\sigma' = (a_1\ b_1\ a_2\ b_3\ a_3) \in H$. Applying the argument presented for $\ell_1 \geq 4$ to this 5-cycle one concludes that H also contains a 3-cycle, finishing this case.

If $\ell_1 = 3$ and $\ell_i < 3$ for $i \neq 1$ then $\sigma^2 \in H$ is a 3-cycle and again we are done.

The final case is that σ is a product of disjoint 2-cycles. Since $\sigma \in H \leqslant A_n$, the number of 2 cycles here is even. Write $\sigma = (a\ b)(c\ d)\sigma_3\ldots\sigma_r$. Conjugation by $(a\ b\ c)$ yields $\sigma' = (b\ c)(a\ d)\sigma_3\ldots\sigma_r \in H$, hence $\sigma\sigma' = (a\ c)(b\ d) \in H$. So H contains the conjugacy class in A_n of $(a\ c)(b\ d)$, which by Theorem VI.1.14 means that *all* products of two disjoint 2-cycles are in H. In particular $(1\,2)(4\,5)\cdot(4\,5)(2\,3) = (1\,2\,3) \in H$. As before, this finishes the proof. ∎

VII.3.8 Remark. The group $A_3 \equiv \mathbb{Z}/3\mathbb{Z}$ simple. But the group A_4 is *not* simple, see Exercise 3

VII.4 Exercises

1. Let G_1 and G_2 be groups let $\varphi : G_1 \to G_2$ be a homomorphism. Show that $\ker(\varphi)$ is normal in G_1.
2. Show that if G is a group and $H \leqslant G$ a subgroup and $N \leqslant G$ normal, then $N \cap H$ is a normal subgroup in H.
3. Let $H \leqslant A_4$ be the subgroup consisting of (1) and all products of two disjoint 2-cycles. Show that H is normal in A_4. Give all elements of A_4/H, and construct a multiplication table for the group A_4/H.
4. Find a normal subgroup in $\mathbb{Z}/2\mathbb{Z} \times A_n$ containing exactly two elements. Prove that $\mathbb{Z}/2\mathbb{Z} \times A_n \ncong S_n$ for $n \neq 2$. This shows that if $H \leqslant G$ is a normal subgroup, then it is not always true that $G \equiv G/H \times H$.
5. Prove that up to isomorphism only one group consisting of 1001 elements exists, as follows: Let G be such a group.
   (a) Show that G contains normal subgroups N_7, N_{11}, and N_{13} with $7, 11$, and 13 elements, respectively.
   (b) Find an injective homomorphism $G \to G/N_7 \times G/N_{11}$.
   (c) Conclude from Theorem VI.4.7 that G is commutative.
   (d) Prove that G contains an element of order 1001, and conclude $G \cong \mathbb{Z}/1001\mathbb{Z}$.
6. Find the subgroups of D_4 and for the normal ones N also D_4/N.
7. Given groups G_1, G_2 with unit elements e_1, e_2, show that $H = G_1 \times \{e_2\}$ is normal in $G_1 \times G_2$, and $(G_1 \times G_2)/H \cong G_2$.
8. Consider $G = \{\left(\begin{smallmatrix} 1 & a \\ 0 & b \end{smallmatrix}\right) \mid b \neq 0\} \subset \mathrm{GL}_2(\mathbb{R})$.
   (a) Show that G is a subgroup of $\mathrm{GL}_2(\mathbb{R})$, but not a normal one.
   (b) Show that $H_1 = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right) \mid b \neq 0\}$ is a subgroup of G, but not a normal one.
   (c) Show that $H_2 = \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{R}\}$ is a normal subgroup in G.
   (d) Verify that $b \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right) H_2$ defines an isomorphism between the multiplicative group $\mathbb{R} \setminus \{0\}$ and G/H_2.
9. Let G be a group and $N \leqslant G$ a normal subgroup. Suppose that H is any subgroup of G containing N.
   (a) Show that N is also normal in H.
   (b) Show that H/N is a subgroup of G/N.
   (c) Show that if $X \leqslant G/N$ is a subgroup, then $Y = \{a \in G \mid aN \in X\}$ is a subgroup of G containing N, and $X = Y/N$.
   (d) Prove that if $Y \leqslant G$ is a subgroup containing N, then Y/N is normal in G/N if and only if Y is normal in G.
10. Fix $k \geq 2$. In a group G consider the subset $H \subset G$ consisting of all products $a_1^k a_2^k \ldots a_n^k$, for $a_1, \ldots, a_n \in G$. Show that H is a normal subgroup in G, and that every element of G/H has order dividing k.
11. In an *abelian* group G put $G = \{a \in G \mid \mathrm{ord}(a) < \infty\}$. Show that H is a normal subgroup in G. Prove that the unit element eH is the only element of G/H whose order is finite. The group H is called the *torsion subgroup* of G, we will encounter it again in Chapter IX.
12. Find an example of a *non-abelian* group G such that $H = \{a \in G \mid \mathrm{ord}(a) < \infty\}$ is not a subgroup of G (e.g., think of real invertible 2×2 matrices).
13. Let $n \geq 5$ and let k be *odd* with $3 \leq k \leq n$.

(a) Let G be a group and let $X \subset G$ be a nonempty subset with the property that also $axa^{-1} \in X$ for all $x \in X$ and all $a \in G$. Prove that

$$H = \{x_1^{\pm 1} \cdot \ldots \cdot x_r^{\pm 1} \mid x_i \in X\}$$

is a normal subgroup in G.

(b) Show that A_n contains an element of order k.

(c) Show that A_5 contains no elements of order 4 or 6.

(d) Use a) to show that every element of A_n can be written as a product of elements of order k.

14. Show that a group of order 48 is not simple. Do the same for groups of order 351.

VIII HOMOMORPHISM AND ISOMORPHISM THEOREMS

Having introduced factor groups in the previous chapter, we now present several tools for determining their structure. The idea is to construct an isomorphism to another related group, whose structure might already be known. To this end, we will first discuss how to construct and describe homomorphisms from a factor group G/H to another group.

VIII.1 Homomorphisms starting from a factor group

We begin with a property of any homomorphism starting from a factor group. Afterwards this property will be used to construct such homomorphisms.

VIII.1.1 Theorem. *Let G and G' be groups, let $H \leqslant G$ be a normal subgroup, and*

$$\varphi\colon G/H \longrightarrow G'$$

a homomorphism. Consider the canonical homomorphism $\pi\colon G \to G/H$ given by $\pi(g) = gH$. Then the composition $\psi = \varphi \circ \pi$ is a homomorphism $G \to G'$.

This homomorphism ψ satisfies $H \subset \ker(\psi)$.

Proof. Check for yourself that any composition of homomorphisms is again a homomorphism. So, in particular, ψ is a homomorphism $G \to G'$.

The second assertion in the theorem follows at once from the definition of ψ and the fact that $\ker(\pi) = H$. ∎

The problem with constructing a homomorphism φ starting from a factor group G/H is that $\varphi(gH)$ has to be independent of the representative g of $gH \in G/H$. An important example of this phenomenon already appeared in Lemma II.3.1. Anyone thoroughly understanding this lemma, will not find the more general situation described below in Criterion VIII.1.2 very challenging, so the reader is advised to recall the lemma and its proof before continuing.

VIII.1.2 Criterion. *Let H be a normal subgroup of a group G, and consider an arbitrary group G'. Constructing a homomorphism $\varphi\colon G/H \to G'$ is done using the following recipe:*

1. *First find a homomorphism $\psi\colon G \to G'$ satisfying $H \subset \ker(\psi)$.*
2. *For ψ as in 1. one has $\psi(g_1) = \psi(g_2)$ for all $g_1, g_2 \in G$ such that $g_1 H = g_2 H$. In other words: the rule $\varphi(gH) = \psi(g)$ yields a well defined map from G/H to G'.*

3. *The map $\varphi\colon G/H \to G'$ as in 2. is a homomorphism and we have $\psi = \varphi\circ\pi$, where π is the canonical homomorphism $G \to G/H$.*

Proof. We first show that any ψ as in 1. satisfies $\psi(g_1) = \psi(g_2)$ in case $g_1H = g_2H$. This follows from the fact that $g_1H = g_2H$ implies $g_2^{-1}g_1 \in H$ by Lemma VII.1.7. Now $H \subset \ker(\psi)$ shows that $g_2^{-1}g_1 \in \ker(\psi)$, so $\psi(g_2^{-1}g_1) = e'$, the unit element of G'. As a consequence $\psi(g_2)^{-1}\psi(g_1) = e'$ and thus $\psi(g_1) = \psi(g_2)$ which is what we wanted to show.

Next we show that the given φ indeed is a homomorphism. Let g_1H, g_2H be elements of G/H. Then $\varphi(g_1H \cdot g_2H) = \varphi(g_1g_2H)$ (by definition of the group law in G/H), and moreover $\varphi(g_1g_2H) = \psi(g_1g_2)$ (this is the definition of φ). Now ψ is a homomorphism, so $\psi(g_1g_2) = \psi(g_1)\psi(g_2)$ which by the definition of φ equals $\varphi(g_1H)\varphi(g_2H)$. We conclude $\varphi(g_1H \cdot g_2H) = \varphi(g_1H)\varphi(g_2H)$, so φ is a homomorphism.

Finally, for arbitrary $g \in G$ we have $(\varphi\circ\pi)(g) = \varphi(gH) = \psi(g)$, so indeed $\psi = \varphi\circ\pi$. ∎

VIII.1.3 Example. We will determine all homomorphisms from $\mathbb{Z}/12\mathbb{Z}$ to $\mathbb{Z}/4\mathbb{Z}$. Using Theorem VIII.1.1 and Criterion VIII.1.2 this boils down to finding all homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}/4\mathbb{Z}$ having $12\mathbb{Z}$ in the kernel. This condition does not provide any restriction. Indeed, if $f\colon \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ is an arbitrary homomorphism and if $n \in 12\mathbb{Z}$, then $n = 12m$ with $m \in \mathbb{Z}$, so in particular $n = 3m+3m+3m+3m$ implying $f(n) = f(3m) + f(3m) + f(3m) + f(3m) = \overline{0}$.

So we simply look for all homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}/4\mathbb{Z}$. Any such homomorphism sends the unit element to the unit element, so 0 to $\overline{0}$. Suppose $\overline{a} \in \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ is the image of 1. This determines the map, because $1 + \ldots + 1$ must be mapped to $\overline{a} + \ldots + \overline{a}$, and the opposite of $1 + \ldots + 1$ to the opposite of $\overline{a} + \ldots + \overline{a}$. Verify yourself that in this way indeed a homomorphism is obtained.

In total we therefore find 4 pairwise different homomorphisms from $\mathbb{Z}/12\mathbb{Z}$ to $\mathbb{Z}/4\mathbb{Z}$. Each of them is completely determined by the image of 1 mod 12. ∎

VIII.1.4 Example. We know that the group D_4 consisting of all symmetries of the square contains precisely 8 elements. We place the square in the plane in such a way that its center is the origin. In this case the 8 symmetries are linear maps $\mathbb{R}^2 \to \mathbb{R}^2$. Together with the identity map, the symmetry 'reflect in the origin' forms a subgroup $H \leqslant D_4$, and we have $H \cong \{\pm 1\}$. It is not hard to verify that H is a normal subgroup of D_4. We now construct an isomorphism $D_4/H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

According to Criterion VIII.1.2 we need to start by constructing a homomorphism from D_4 to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A homomorphism $D_4 \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ is induced by the determinant. Furthermore, the square has two diagonals, and every element of D_4 permutes these two. This defines a second homomorphism $f\colon D_4 \to S_2 \cong \mathbb{Z}/2\mathbb{Z}$. The pair $\psi = (\det, f)$ is the requested homomorphism

$$\psi\colon D_4 \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \;:\; \sigma \mapsto (\det(\sigma), f(\sigma)) \in \pm 1 \times S_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Note that ψ is surjective (find explicit elements in D_4 that are mapped to each of the elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by yourself). The kernel of ψ consists by definition of all symmetries fixing the two diagonals and having determinant 1. So the kernel consists of the rotations ± 1, in other words it is precisely our subgroup H. So the condition $H \subset \ker(\psi)$ in Criterion VIII.1.2 is satisfied. One concludes that a homomorphism $\varphi\colon D_4/H \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ exists, defined by $\varphi(\sigma H) = \psi(\sigma)$. Since $\psi(\sigma)$ ranges over all elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, it follows that φ is surjective. The number of elements in D_4/H equals $[D_4 : H] = \#D_4/\#H = 8/2 = 4$ which is also the number of elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since φ is surjective this implies that φ is a bijection, and therefore it is an isomorphism. So $D_4/H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, as asserted. ∎

VIII.2 Isomorphism theorems for factor groups

The most important frequently used rule for dealing with factor groups reads as follows.

VIII.2.1 Theorem. (Homomorphism theorem) *If $\psi\colon G \to G'$ is a homomorphism of groups, then $H := \ker(\psi)$ is a normal subgroup of G and we have*

$$G/H \cong \psi(G) \leqslant G'.$$

In particular, if ψ is surjective, then one has $G/H \cong G'$.

Proof. The fact that H is a normal subgroup is already part of Theorem VII.2.11. Moreover, by Criterion VIII.1.2, setting $\varphi(gH) := \psi(g)$ results in a well defined homomorphism $\varphi\colon G/H \to G'$.

We determine the kernel of φ. Note that $gH \in \ker(\varphi)$ precisely when $\varphi(gH)$ is the unit element $e' \in G'$. Now $\varphi(gH) = \psi(g)$ equals e' if and only if $g \in \ker(\psi) = H$. Moreover $g \in H$ is equivalent to $gH = eH$, i.e., gH is the unit element in G/H. The conclusion is that $\ker(\varphi)$ consists of only the unit element in G/H. Theorem III.3.6 therefore implies that φ is injective.

Injectivity of φ yields that G/H is isomorphic to the image of φ, and by definition this equals the image of ψ. So $G/\ker(\psi) \cong \psi(G)$, as required. In case ψ is surjective we have $\psi(G) = G'$ and hence $G/\ker(\psi) \cong G'$. ∎

VIII.2.2 Example. The determinant is a surjective homomorphism from $\mathrm{GL}_n(\mathbb{R})$ to the multiplicative group $(\mathbb{R}\setminus\{0\}, \cdot, 1)$ by Example III.3.2. The kernel of the determinant is $\mathrm{SL}_n(\mathbb{R})$, so we get

$$\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R}) \cong \mathbb{R}\setminus\{0\}$$

by Theorem VIII.2.1 —∎

VIII.2.3 Example. The complex numbers $a + bi$ satisfying $a^2 + b^2 = 1$ form a subgroup $\mathbf{T}$ of the multiplicative group $(\mathbb{C}\setminus\{0\}, \cdot, 1)$, see Exercise III.10. This subgroup is isomorphic to the factor group $\mathbb{R}/\mathbb{Z}$. Indeed, $x \mapsto e^{2\pi i x}$ defines a surjective homomorphism $\mathbb{R} \to \mathbf{T}$ with kernel $\mathbb{Z}$. —∎

The next two isomorphism theorems for factor groups are in fact consequences of Theorem VIII.2.1.

VIII.2.4 Theorem. (First isomorphism theorem) *Consider a group G, an arbitrary subgroup $H \leqslant G$, and a normal subgroup $N \leqslant G$. Then*

1. *$HN = \{hn \mid h \in H \text{ and } n \in N\}$ is a subgroup of G.*
2. *N is a normal subgroup of HN.*
3. *$H \cap N$ is a normal subgroup of H.*
4. *$H/(H \cap N) \cong HN/N$.*

Proof. 1: By Theorem III.2.3 we have to check the three conditions (H1, H2, H3). H1: from $e = e \cdot e$ and $e \in H, e \in N$ we see $e \in HN$. H3: for arbitrary $h \in H$ and $n \in N$ we know $hn^{-1}h^{-1} \in N$ since N is normal in G. So $(hn)^{-1} = n^{-1}h^{-1} = h^{-1}\cdot(hn^{-1}h^{-1})$ is in HN. Finally H2: for $h_1, h_2 \in H$ and $n_1, n_2 \in N$ we see $h_2^{-1}n_1h_2 \in N$ since $N \leqslant G$ is normal. Therefore $(h_1n_1)\cdot(h_2n_2) = h_1h_2(h_2^{-1}n_1h_2)n_2 \in HN$. So indeed $HN \subset G$ is a subgroup.
2: Since $e \in H$, it follows that $N = eN \subset HN$. Since N is a group, it is therefore a

subgroup of HN. We have $gNg^{-1} = N$ for all $g \in G$, so certainly for those $g \in G$ which are in HN. Hence N is normal in HN.
3 and 4: define $\psi\colon H \to G/N$ by $\psi(h) := hN \in G/N$. This is the restriction to H of the canonical homomorphism $G \to G/N$, so ψ is a homomorphism. Note $h \in \ker(\psi)$ if and only if $hN = N$, i.e. if and only if $h \in N$. So $\ker(\psi) = H \cap N$, which implies that $H \cap N$ is normal in H by Theorem VII.2.11. Theorem VIII.2.1 tells us that $H/(H \cap N)$ is isomorphic to the image of ψ. So our argument is complete if we have shown $\psi(H) = HN/N$. This is straightforward: an element of $\psi(H)$ can be written as $hN \in G/N$, and here $h \in H \subset HN$, so this element is in HN/N. Vice versa, an element in HN/N can be written as hnN with $h \in H$ and $n \in N$. Now $nN = N$ hence $hnN = hN$, which is the image of $h \in H$ under ψ. ∎

VIII.2.5 Example. Take $G = \mathbb{Z}$, $n, h \in \mathbb{Z}$ and $H = h\mathbb{Z}$ and $N = n\mathbb{Z}$. The group law in $\mathbb{Z}$ is 'addition'; therefore Theorem VIII.2.4 implies that $h\mathbb{Z}/(h\mathbb{Z} \cap n\mathbb{Z}) \cong (h\mathbb{Z}+n\mathbb{Z})/n\mathbb{Z}$. We analyse this a bit further. Note that $h\mathbb{Z} \cap n\mathbb{Z}$ consists of all integers which are both divisible by h and by n. Corollary I.2.10 5. asserts that these are precisely the multiples of $\operatorname{lcm}(h,n)$. So $h\mathbb{Z} \cap n\mathbb{Z} = \operatorname{lcm}(h,n)\mathbb{Z}$. Moreover Theorem I.1.12 implies $h\mathbb{Z} + n\mathbb{Z} = \gcd(h,n)\mathbb{Z}$. We conclude

$$h\mathbb{Z}/\operatorname{lcm}(h,n)\mathbb{Z} \cong \gcd(h,n)\mathbb{Z}/n\mathbb{Z}.$$

In the special case $\gcd(h,n) = 1$ this says $h\mathbb{Z}/hn\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$. Incidentally, this also holds for $\gcd(h,n) \neq 1$, as can (for example) be shown using Theorem VIII.2.1. ▬

VIII.2.6 Example. Consider $N := \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ which is a subgroup of $G = S_4$, and $H =$ all permutations in S_4 fixing the integer 4 (thus $S_3 \cong H \leqslant S_4$). Note that N is normal in G. We have $HN = S_4$, since both H and N are subgroups of HN, so $\#HN$ is divisible by both $\#H = 4$ and $\#N = 6$. Hence $12 \mid \#HN$. This implies that $[S_4 : HN] = 1$ or $= 2$. In particular, HN is normal in S_4. Now S_4/HN has at most two elements, so this factor group is abelian, implying $A_4 \subset HN$. Noting that HN contains odd permutations as well, it follows that $HN = S_4$. (This may be verified in numerous other ways, but the argument above illustrates a number of techniques we now have at our disposal.) Observing that $H \cap N = \{(1)\}$, Theorem VIII.2.4 implies

$$S_4/H = HN/H \cong H/(H \cap N) = S_3/\{(1)\} = S_3.$$

▬

Part of the next result we encountered in Exercise 9 of Chapter VII.

VIII.2.7 Theorem. (Second isomorphism theorem) *Consider a group G and a normal subgroup $N \leqslant G$.*

1. *Every normal subgroup in G/N has the form H/N, with H a normal subgroup in G containing N.*
2. *If $N \subset H$ for some normal subgroup H in G, then $(G/N)/(H/N) \cong G/H$.*

Proof. For (1) we refer to Exercise 9 in Chapter VII.
(2): Consider the canonical homomorphism $\pi\colon G \to G/H$. We have $N \subset H$ and by Theorem VII.2.9 $H = \ker(\pi)$, so $N \subset \ker(\pi)$. Hence applying Criterion VIII.1.2 one concludes that $\psi(gN) = \pi(g) = gH$ defines a homomorphism $\psi\colon G/N \to G/H$. This homomorphism ψ is surjective because π is surjective. Moreover $gN \in \ker(\psi)$ precisely when $\psi(gN) = gH = eH$, so $\ker(\psi)$ consists of all classes gN with $g \in H$. We conclude that $\ker(\psi) = H/N$. From Theorem VIII.2.1 we now deduce

$$(G/N)/(H/N) = (G/N)/\ker(\psi) \cong \psi(G/N) = G/H,$$

which is what we wanted to prove. ∎

VIII.2.8 Example. The residue classes $2a$ mod 6 for $a \in \mathbb{Z}$ form a normal subgroup of $\mathbb{Z}/6\mathbb{Z}$. This is precisely $2\mathbb{Z}/6\mathbb{Z}$, and Theorem VIII.2.7 applied to $G = \mathbb{Z}$ and $N = 6\mathbb{Z}$ and $H = 2\mathbb{Z}$ says that $(\mathbb{Z}/6\mathbb{Z})/(2\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. ▬

VIII.2.9 Remark. The homomorphism theorem (Theorem VIII.2.1) is sometimes called the first isomorphism theorem in the literature, whereas the results which we refer to as the first and second isomorphism theorem, respectively, are called the second and third isomorphism theorem, respectively. There is no consensus on the terminology.

VIII.3 Exercises

1. Determine all homomorphisms from $\mathbb{Z}/4\mathbb{Z}$ to $\mathbb{Z}/6\mathbb{Z}$.
2. Prove that $\mathbb{C}/\mathbb{Z}$ is isomorphic to the multiplicative group $(\mathbb{C}\setminus\{0\},\cdot,1)$.
3. Show that for $n,h\in\mathbb{Z}$ with $h\neq 0$ we have $h\mathbb{Z}/hn\mathbb{Z}\cong\mathbb{Z}/n\mathbb{Z}$.
4. For $N:=\{(1),(1\,2)(3\,4),(1\,3)(2\,4),(1\,4)(2\,3)\}\leqslant S_4$, show that $(S_4/N)/(A_4/N)\cong\mathbb{Z}/2\mathbb{Z}$.
5. Show that if $k|N$ then $(\mathbb{Z}/N\mathbb{Z})/(k\mathbb{Z}/N\mathbb{Z})\cong\mathbb{Z}/k\mathbb{Z}$. Is the condition $k|N$ necessary?
6. Let $n,m\in\mathbb{Z}$ be both positive. In the group D_{nm} let ρ denote the counter clockwise rotation by $2\pi/nm$ and let $\sigma=$ denote 'reflection in the x-axis'. Then we know that $\sigma\rho\sigma=\rho^{-1}$. Consider $H:=\{\mathrm{id},\sigma\}$ and $N:=\{\mathrm{id},\rho^m,\rho^{2m},\ldots,\rho^{(n-1)m}\}$.
   (a) Show that H,N are subgroups of D_{nm} and tjat N is normal in D_{nm}.
   (b) Prove that $HN\cong D_n$.
   (c) Prove that $D_m\cong D_{nm}/N$.
7. Let G be a group and $H_1,H_2\leqslant G$ normal subgroups. Define $\psi\colon G\to G/H_1\times G/H_2$ by $\psi(g)=(gH_1,gH_2)$.
   (a) Show that ψ is a homomorphism and $H_1\cap H_2$ is a normal subgroup in G.
   (b) Prove that $G/(H_1\cap H_2)$ is isomorphic to a subgroup of $G/H_1\times G/H_2$.
   (c) Use (b) to obtain a new proof of the Chinese Remainder Theorem.
8. We will show that for $n\geq 5$ the only normal subgroup in S_n different from $\{(1)\}$ or all of S_n, is A_n. Let N be such a nontrivial normal subgroup in S_n.
   (a) Use that A_n is simple and show that $A_n\subset N$ or $N\cap A_n=\{(1)\}$.
   (b) Show that in case $A_n\subset N$ it follows that $N=A_n$.
   (c) Show that if $N\neq\{(1)\}$ and $N\cap A_n=\{(1)\}$, then $NA_n=S_n$ and $S_n/N\cong A_n$.
   (d) Conclude in the situation of (c) that $\#N=2$, and prove this contradicts the assumption that $N\leqslant S_n$ is normal.

IX FINITELY GENERATED ABELIAN GROUPS

In this chapter we mainly consider abelian groups. Our main goal will be to present a complete description of all so-called finitely generated abelian groups. Some of the techniques used in this chapter are similar to techniques familiar from linear algebra.

IX.1 Finitely generated groups

IX.1.1 Definition. A group G is called *finitely generated* if there exist elements $g_1,\dots,g_n \in G$ with the following property: Every $g \in G$ can be written as

$$g = g_{i_1}^{\pm 1} \cdot \ldots \cdot g_{i_t}^{\pm 1}$$

with indices $1 \le i_j \le n$ (note that it is allowed here that $i_k = i_\ell$, in other words any g_i can be used multiple times).

So a group G is finitely generated if it has a finite set of *generators* $g_1,\dots,g_n$, which means that every element of G can be written as a product of the generators and their inverses.

IX.1.2 Example.

1. Every finite group G is finitely generated, since in this case the set of all elements in G as generators.
2. The group $\mathbb{Z}^r = \mathbb{Z} \times \ldots \times \mathbb{Z}$ (the product of r copies of $\mathbb{Z}$) is finitely generated. Namely, take $e_1 = (1,0,\dots,0)$, $e_2 = (0,1,0,\dots,0),\dots\dots,e_r = (0,\dots,0,1) \in \mathbb{Z}^r$. An arbitrary $(m_1,\dots,m_r) \in \mathbb{Z}^r$ can be written as $m_1e_1 + \ldots + m_re_r$. Moreover the integers m_j here are uniquely determined; the group $\mathbb{Z}^r$ is therefore an example of a so-called *free abelian group*, with basis $e_1,\dots,e_r$. The statement, as well as the terminology are reminiscent of the theory of vector spaces in linear algebra. Such groups will be treated in more detail below.
3. The additive group $(\mathbb{Q},+,0)$ is *not* finitely generated. Namely, given arbitrary $g_1,\dots,g_n \in \mathbb{Q}$, the finite sum $\pm g_{i_1} \pm \ldots \pm g_{i_t}$ can be written as a/b with $a \in \mathbb{Z}$ and b equal to the least common multiple of the denominators of $g_1,\dots,g_n$. Hence a number $c/d \in \mathbb{Q}$ with $c,d \in \mathbb{Z}$ and $\gcd(c,d) = 1$ and d larger than this least common multiple can not be expressed as a sum of $\pm g_i$'s. Therefore no finite set $\{g_1,\dots,g_n\} \subset \mathbb{Q}$ generates all of $\mathbb{Q}$.
4. It is possible that a (non-abelian) group G is finitely generated whereas some subgroup $H \leqslant G$ is not. A nice example of this phenomenon is described in

the paper B.L. van der Waerden, *Example d'un groupe avec deux générateurs, contenant un sous-groupe commutatif sans système fini de générateurs,* which appeared in the journal Nieuw Archief voor Wiskunde, Vol. 23 (1951), p. 190.
In a similar (and in fact much easier) way it is possible that a finitely generated group is generated by elements of finite order, and yet the group contains elements of infinite order. For example we let σ_1 denote the reflection in the x-axis and σ_2 the reflection in the line with equation $y = ax$; both are elements of the infinite dihedral group D_∞. Then $\sigma_2\sigma_1$ is rotation by an angle α with $\tan(\alpha/2) = a$. For suitable a this rotation has infinite order. Clearly both σ_1 and σ_2 have order 2.

—■

In the remainder of this chapter we restrict ourselves to abelian groups. The group law will be denoted by $+$.

IX.1.3 Theorem. *Any finitely generated abelian group $(A,+,0)$ is isomorphic to a factor group $\mathbb{Z}^n/H$ for some subgroup $H \leqslant \mathbb{Z}^n$.*

Proof. Let the set $\{a_1,\ldots,a_n\}$ generate the group A. Define

$$\varphi \;:\; \mathbb{Z}^n \longrightarrow A$$

by $\varphi(m_1,\ldots,m_n) = m_1a_1+\ldots+m_na_n$. It is easy to verify that φ is a homomorphism. Moreover φ is surjective because $a_1,\ldots,a_n$ generate A, so every element of A is an additive combination of the elements $\pm a_i$. Since A is abelian, here the order of the sequence of $\pm a_i$'s is irrelevant in this case. So any $a \in A$ can be written as $a = n_1a_1 + \ldots + n_na_n$ which is the image of $(n_1,\ldots,n_n)$ under φ.

Put $H = \ker(\varphi)$. This is a subgroup of $\mathbb{Z}^n$. Theorem VIII.2.1 therefore implies

$$\mathbb{Z}^n/H = \mathbb{Z}^n/\ker(\varphi) \cong \varphi(\mathbb{Z}^n) = A.$$

■

IX.2 Subgroups of free abelian groups

Theorem IX.1.3 shows that describing all finitely generated abelian groups boils down to describing all subgroups $H \leqslant \mathbb{Z}^n$ and the corresponding factor groups $\mathbb{Z}^n/H$. The case $n = 1$ was discussed in Example III.2.7; here $H = m\mathbb{Z}$ for some $m \geq 0$. So $\mathbb{Z}/H \cong \mathbb{Z}$ in case $m = 0$ and $\mathbb{Z}/H = (0)$ if $m = 1$, and $\mathbb{Z}/H = \mathbb{Z}/m\mathbb{Z}$ in general.

IX.2.1 Theorem. *If $H \leqslant \mathbb{Z}^n$ is a subgroup then $H \cong \mathbb{Z}^k$ for some k with $0 \leq k \leq n$.*

Proof. We use mathematical induction with respect to n. The case $n = 0$ is trivial, and the case $n = 1$ follows from Example III.2.7; here $H = m\mathbb{Z}$ with $m \geq 0$. For $m = 0$ we get $H = (0) \cong \mathbb{Z}^0$, and for $m > 0$ we have $\mathbb{Z} \cong m\mathbb{Z}$, an explicit isomorphism is given by multiplication by m.

As induction hypothesis, assume the theorem holds for $n \geq 1$. Let $H \leqslant \mathbb{Z}^{n+1}$ be a subgroup. Define

$$\pi \;:\; \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z}\,;\; (m_1,\ldots,m_{n+1}) \mapsto m_{n+1}.$$

This is a homomorphism with kernel all sequences $(m_1,\ldots,m_{n+1}) \in \mathbb{Z}^{n+1}$ such that $m_{n+1} = 0$. We can therefore identify this kernel with $\mathbb{Z}^n$. Since $H \leqslant \mathbb{Z}^{n+1}$ is a subgroup, so is $H \cap \ker(\pi) \leqslant \mathbb{Z}^n$. Hence the induction hypothesis implies $H \cap \ker(\pi) \cong \mathbb{Z}^k$ for some k with $0 \leq k \leq n$.

Since $H \leqslant \mathbb{Z}^{n+1}$ is a subgroup, so is $\pi(H) \leqslant \mathbb{Z}$. Hence $\pi(H) = m\mathbb{Z}$ for some $m \geq 0$. If $m = 0$, then $\pi(H) = (0)$ so $H \leqslant \ker(\pi)$ which implies $\mathbb{Z}^k \cong H \cap \ker(\pi) = H$. So in this case the proof is complete. From now on we assume $m \neq 0$. Since $m \in m\mathbb{Z} = \pi(H)$, we have $h_{k+1} \in H$ with $\pi(h_{k+1}) = m$. Choose an isomorphism $\mathbb{Z}^k \cong H \cap \ker(\pi)$ and let $h_1, \ldots, h_k \in H \cap \ker(\pi)$ denote the images of $(1,0,\ldots,0), \ldots\ldots, (0,\ldots,0,1)$ under this isomorphism. We will show that

$$\psi : \mathbb{Z}^{k+1} \longrightarrow H \text{ defined by } \psi(m_1,\ldots,m_{k+1}) := m_1h_1 + \ldots + m_{k+1}h_{k+1}$$

is an isomorphism. Clearly ψ is a homomorphism and $\mathbb{Z}^{k+1}$ is mapped by ψ into H. We first show that ψ is surjective. Take an arbitrary $h \in H$. Since $\pi(h) \in \pi(H) = m\mathbb{Z}$, we have $\pi(h) = m\ell$ for some $\ell \in \mathbb{Z}$. From this one deduces

$$\pi(h - \ell h_{k+1}) = \pi(h) - \pi(\ell h_{k+1}) = m\ell - \ell m = 0,$$

so $h - \ell h_{k+1} \in \ker(\pi) \cap H$. Using our isomorphism $\mathbb{Z}^k \cong \ker(\pi) \cap H$ we find integers $\ell_1, \ldots, \ell_k$ with $h - \ell h_{k+1} = \ell_1 h_1 + \ldots + \ell_k h_k$. This shows $h = \psi(\ell_1, \ldots, \ell_k, \ell)$, so ψ is surjective.

Next we show that ψ is injective. By Theorem III.3.6 it suffices to verify that $\ker(\psi) = \{(0,\ldots,0)\} \leqslant \mathbb{Z}^{k+1}$. Let $(m_1,\ldots,m_{k+1}) \in \ker(\psi)$. Then $m_1h_1 + \ldots + m_{k+1}h_{k+1} = 0 \in H$, so $m_1h_1 + \ldots + m_kh_k = -m_{k+1}h_{k+1}$. From $h_1, \ldots, h_k \in \ker(\pi)$ it follows that $-m_{k+1}h_{k+1} \in \ker(\pi)$. Hence $0 = \pi(-m_{k+1}h_{k+1}) = -m_{k+1}m$. Our assumption $m \neq 0$ yields $m_{k+1} = 0$, so $m_1h_1 + \ldots + m_kh_k = -m_{k+1}h_{k+1} = 0$. Using $\mathbb{Z}^k \cong H \cap \ker(\pi)$ we conclude $(m_1,\ldots,m_k) = (0,\ldots,0) \in \mathbb{Z}^k$. So $m_1 = \ldots = m_k = m_{k+1} = 0$, which shows that ψ is injective.

So ψ is an isomorphism, which finishes the induction argument. ∎

IX.2.2 Remark. The argument above repeatedly uses the fact that for a abelian group H one has $H \cong \mathbb{Z}^k$ if and only if $h_1, \ldots, h_k \in H$ exist such that every $h \in H$ can be written in a *unique* way as $h = m_1h_1 + \ldots + m_kh_k$. A group H having this property is called a free abelian group (with basis $h_1, \ldots, h_k$). According to Theorem IX.2.1, any subgroup of a finitely generated free abelian group is itself a finitely generated free abelian group. In fact free abelian groups behave quite similarly to vector spaces in many ways. Further generalizations of vector spaces are discussed in the course Advanced Algebraic Structures.

IX.2.3 Example. Consider $H \leqslant \mathbb{Z}^3$ given by

$$H = \{(a,b,c) \in \mathbb{Z}^3 \mid a + 2b + 3c \equiv 0 \bmod 6\}.$$

It is not hard to verify that H is a subgroup of $\mathbb{Z}^3$ (for example: H is the kernel of the homomorphism $\mathbb{Z}^3 \to \mathbb{Z}/6\mathbb{Z}$ given by $(a,b,c) \mapsto a+2b+3c \bmod 6$). By Theorem IX.2.1 and Remark IX.2.2 there exist $r \in \{0,1,2,3\}$ and $h_1, \ldots, h_r \in H$ such that H is the free abelian group with basis $h_1, \ldots, h_r$. We now determine such $r, h_1, \ldots, h_r$ by the method used in the proof of Theorem IX.2.1.

Let $\pi_i : \mathbb{Z}^3 \to \mathbb{Z}$ be the projection on the ith coordinate. We have $\pi_3(H) = \mathbb{Z}$ since $(1,1,1) \in H$ and hence $1 \in \pi_3(H)$; a subgroup of $\mathbb{Z}$ containing 1 equals $\mathbb{Z}$. The proof of Theorem IX.2.1 now shows that $(1,1,1)$ together with a basis for $\ker(\pi_3) \cap H$ yields a basis of H. By definition

$$\ker(\pi_3) \cap H = \{(a,b,0) \mid a + 2b \equiv 0 \bmod 6\}.$$

We find $\pi_2(\ker(\pi_3) \cap H) = \mathbb{Z}$, because $(4,1,0) \in \ker(\pi_3) \cap H$ and $\pi_2(4,1,0) = 1$. So a basis for $\ker(\pi_3) \cap H$ consists of $(4,1,0)$ together with a basis for $\ker(\pi_2) \cap \ker(\pi_3) \cap H$. We have

$$\ker(\pi_2) \cap \ker(\pi_3) \cap H = \{(a,0,0) \mid a \equiv 0 \bmod 6\} = \mathbb{Z}(6,0,0),$$

hence

$$H = \mathbb{Z} \cdot (6,0,0) + \mathbb{Z} \cdot (4,1,0) + \mathbb{Z} \cdot (1,1,1).$$

∎

We will now show that given any subgroup $H \leqslant \mathbb{Z}^n$, there is only one integer k with $H \cong \mathbb{Z}^k$, and moreover $0 \leq k \leq n$. This follows directly from Theorem IX.2.1 and the next result.

IX.2.4 Theorem. *If $\mathbb{Z}^k \cong \mathbb{Z}^\ell$, then $k = \ell$.*

Proof. Suppose $\mathbb{Z}^k \cong \mathbb{Z}^\ell$ and choose an isomorphism $\mathbb{Z}^k \to \mathbb{Z}^\ell$. Consider the composition $\mathbb{Z}^k \cong \mathbb{Z}^\ell \to \mathbb{Z}^\ell/2\mathbb{Z}^\ell$. Here the second map is the canonical homomorphism to a factor group, and $2\mathbb{Z}^\ell = 2\mathbb{Z} \times \ldots \times 2\mathbb{Z}$. This composition is a surjective homomorphism, and its kernel is $2\mathbb{Z}^k$. Hence Theorem VIII.2.1 implies $\mathbb{Z}^k/2\mathbb{Z}^k \cong \mathbb{Z}^\ell/2\mathbb{Z}^\ell$.

For any $m \geq 0$ one finds that $\mathbb{Z}^m/2\mathbb{Z}^m \cong (\mathbb{Z}/2\mathbb{Z})^m$, since the homomorphism $\mathbb{Z}^m \to (\mathbb{Z}/2\mathbb{Z})^m$ given by $(n_1, \ldots, n_m) \mapsto (n_1 \bmod 2, \ldots, n_m \bmod 2)$ is surjective and has kernel $2\mathbb{Z}^m$; now apply Theorem VIII.2.1.

In our situation, combining the above arguments we find $(\mathbb{Z}/2\mathbb{Z})^k \cong (\mathbb{Z}/2\mathbb{Z})^\ell$. These groups have 2^k and 2^ℓ elements, respectively and therefore $k = \ell$. ∎

IX.2.5 Corollary. *If $H \leqslant \mathbb{Z}^n$ is a subgroup then a unique integer k exists with $H \cong \mathbb{Z}^k$ (and this k satisfies $0 \leq k \leq n$).*

Proof. By Theorem IX.2.1 k exists. If both k_1 and k_2 have the desired property then $\mathbb{Z}^{k_1} \cong H \cong \mathbb{Z}^{k_2}$, hence by Theorem IX.2.4 it follows that $k_1 = k_2$. ∎

IX.3 The structure of finitely generated abelian groups

We start by stating the main theorem concerning finitely generated abelian groups.

IX.3.1 Theorem. (Structure theorem (or fundamental theorem) for finitely generated abelian groups) *For any finitely generated abelian group there exist a unique integer $r \geq 0$ and a unique (possibly empty) finite sequence $(d_1, \ldots, d_m)$ of integers $d_i > 1$ satisfying $d_m | d_{m-1} | \ldots | d_1$, such that*

$$A \cong \mathbb{Z}^r \times \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_m\mathbb{Z}.$$

IX.3.2 Definition. Given a finitely generated abelian group A, the integer r mentioned in Theorem IX.3.1 is called the *rank* of A. The integers $d_1, \ldots, d_m$ are called the *elementary divisors* of A.

The structure theorem is an immensely powerful result, stating that, up to isomorphism, any finitely generated abelian group is determined uniquely by its rank and elementary divisors. Its proof will take up most of the rest of this section; along the way, we will also discuss how to compute r and $d_1, \ldots, d_m$.

IX.3.3 Example.

1. Any subgroup $H \leqslant \mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^k$ for a unique k by Corollary IX.2.5. In particular, this implies that H is finitely generated, and Theorem IX.3.1 implies that $\operatorname{rank}(H) = k$ and $d_1, \ldots, d_m = \varnothing$.
2. The finite abelian group $A = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ clearly has $\operatorname{rank}(A) = 0$ and elementary divisors $(d_1, d_2) = (12, 2)$. Indeed, applying the Chinese Remainder Theorem (see Example III.3.2 4)

$$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

∎

The most important ingredient in the proof of Theorem IX.3.1 reads as follows.

IX.3.4 Theorem. *Given a subgroup $H \leqslant \mathbb{Z}^n$ with $H \neq (0)$, there exists a basis $f_1,\dots,f_n$ for $\mathbb{Z}^n$, an integer k with $1 \leq k \leq n$ and a sequence of integers $(d_1,\dots,d_k)$ with $d_i > 0$ and $d_k | d_{k-1} | \dots | d_1$ such that $d_1 f_1,\dots,d_k f_k$ is a basis of H.*

Proof. Take a basis $e_1,\dots,e_n$ for $\mathbb{Z}^n$ (say, the standard one) and a basis $g_1,\dots,g_k$ for H (it exists by Theorem IX.2.1). Then $g_i = a_{1i}e_1 + \dots + a_{ni}e_n$ $(i = 1,\dots,k)$ for certain $a_{ij} \in \mathbb{Z}$. The integers a_{ij} form a matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} \end{pmatrix}.$$

Each pair of bases $(\beta = (e_1,\dots,e_n), \gamma = (g_1,\dots,g_k)$ yields in this way an $n \times k$ matrix with integer coefficients, expressing how the basis γ is given in terms of the basis β. Replacing the basis β or the basis γ by a different one a different matrix is obtained. Our aim is to change these bases into a β' for $\mathbb{Z}^n$ and a γ' for H such that the resulting matrix is

$$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_k \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

for integers $(d_1,\dots,d_k)$ with $d_i > 0$ and $d_k | d_{k-1} | \dots | d_1$.

The next algorithm brings our initial matrix into a matrix of the desired form in finitely many steps. After presenting the algorithm, we will show that indeed it corresponds to a change of the two bases.

Algorithm.

step 1 If A is the zero matrix, we are done. If not, take (i,j) such that $|a_{ij}| > 0$ is minimal. Interchange the first and the i-th row as well as the first and the j-th column. In the new matrix, $|a_{11}| > 0$ is the minimal nonzero absolute value of an entry. We now try to make the remaining entries in the first column zero.

step 2 If an integer a_{i1} in the first column (for $i \neq 1$) is nonzero, add a suitable multiple of the first row to the i-th row such that a_{i1} is replaced by an integer r with $0 \leq r < |a_{11}|$. If $r \neq 0$, then interchanging the i-th and the first row. Repeat this step until the first column only has a nonzero entry in place $(1,1)$.

step 3 Analogously, make the remaining entries in the first row equal to zero.

step 4 We now make sure that all entries in the matrix are multiples of a_{11}, as follows. If $a_{11} \nmid a_{ij}$, replace the i-th row by the sum of the i-th and the first (this only changes one entry in the first column), and add a suitable multiple of the first column to the j-th. This yields a_{ij} with $0 \leq a_{ij} < |a_{11}|$. It is $\neq 0$, since otherwise a_{ij} would have been divisible by a_{11}. Now start all over at step 1 with the new matrix. The new a_{11} obtained in step 1 is in absolute value strictly smaller than the old one, hence after finitely many steps indeed all a_{ij} are multiples of a_{11}.

step 5 Apply steps 1 to 4 to the matrix obtained from the one found so far by deleting the first row and the first column. All entries of this smaller matrix are multiples of the a_{11} constructed above, and this property remains true during the steps. So at the end, the smaller matrix has in its top left corner an integer a_{22} which is a multiple of a_{11} and the remaining entries in its first row and

column are 0. Moreover a_{22} divides a_{ij} for all $i,j \geq 3$. Continuing in this way results in a matrix with $a_{ij} = 0$ if $i \neq j$ and $a_{11}|a_{22}|\ldots|a_{kk}$.

step 6 Finally, multiply rows by ± 1 and put the first k vectors in the two bases in the reverse order to obtain a matrix as desired.

We now show that the changes made in the algorithm to the initial matrix correspond to changes of a basis for either $\mathbb{Z}^n$ or H. To this end, we describe some ways of changing a basis, and we explain the effect it has on the matrix.

1. Interchange the j-th and the k-th basis vector in the basis for $\mathbb{Z}^n$.
   Obviously this results in a new basis for $\mathbb{Z}^n$. An element $\sum_i a_i e_i \in \mathbb{Z}^n$ is given as $a_1e_1 + \ldots + a_k e_k + \ldots + a_j e_j + \ldots + a_n e_n$ in terms of the new basis. Hence this corresponds to interchanging the j-th and the k-th row in the matrix.
2. Interchange the j-th and the k-th basis vector in the basis for H. The effect on the matrix is that the j-th and the k-th column are interchanged.
3. Replace the j-th basis vector e_j of $\mathbb{Z}^n$ by its opposite $-e_j$.
   This yields of course a new basis of $\mathbb{Z}^n$. With respect to the new basis an element of $\mathbb{Z}^n$ has as j-th coordinate -1 times the old j-th coordinate. Hence the effect on our matrix is that all entries in the j-th row are multiplied by -1.
4. Replace the j-th basis vector of the given basis for H by its opposite. The effect on the matrix is that all integers in the j-th column are multiplied by -1.
5. Let $a \in \mathbb{Z}$ and $i \neq j$ and replace the basis vector e_i by $e_i' = e_i - ae_j$ in the basis for $\mathbb{Z}^n$.
   This results in a new basis for $\mathbb{Z}^n$. Indeed, the map $\mathbb{Z}^n \to \mathbb{Z}^n$ given by
   $$\sum_k a_k e_k \mapsto a_1e_1 + \ldots + a_i e_i + \ldots + (a_i a - a_j)e_j + \ldots + a_n e_n$$
   is an isomorphism of groups (check this yourself!), and this map sends the vector $\{e_1,\ldots,e_i,\ldots,e_n\}$ to $\{e_1,\ldots,e_i',\ldots,e_n\}$.
   Since
   $$\sum_k a_k e_k = a_1e_1 + \ldots + a_i(e_i - ae_j) + \ldots + (a_j + a_i a)e_j + \ldots + a_n e_n,$$
   it follows that the effect of this change of basis on the matrix is that the j-th row is replaced by the sum of the j-th row and a times the i-th row.
6. Finally, in the basis for H one can replace basis vector g_i by $g_i - ag_j$ in an analogous way. Similar to the above, the effect on the matrix is that the j-th column is replaced by the j-th column plus a times the i-th column.

The conclusion from the base changes described here is that if the $n \times k$ matrix A expresses how a basis of H is represented with respect to a basis of $\mathbb{Z}^n$, and if the matrix B is obtained from A by repeatedly executing the following steps:

1. interchange two rows or two columns in the given matrix;
2. multiply a row or a column by -1 in the given matrix;
3. add a times a *different* row/column to a given row/column in the given matrix;

then also B expresses how some basis for H is given in terms of some basis for $\mathbb{Z}^n$. Since the changes made in the algorithm are exactly of this form, his completes the proof of Theorem IX.3.4. ∎

IX.3.5 Remark. In fact the procedure presented above can be used in a more general context. If $H \leqslant \mathbb{Z}^n$ is given as $H = \mathbb{Z}g_1 + \ldots + \mathbb{Z}g_k$ for certain $g_i \in \mathbb{Z}^n$, without assuming that the g_i are a basis for H, a matrix can be produced from the generators g_i in exactly the same way. Applying the algorithm to this matrix changes it to a new one whose only nonzero entries lie on the diagonal (unless $H = \{0\}$). The

nonzero columns of the resulting matrix describe a basis for H, expressed in terms of some basis for $\mathbb{Z}^n$. Hence, given a finite set of generators for some subgroup of $\mathbb{Z}^n$, this provides a way to obtain on the one hand a new proof of Theorem IX.2.1 for this subgroup and, on the other hand, to construct a basis for this subgroup.

Using Theorem IX.3.4 we will now show the existence of $r, d_1, \ldots, d_m$ as given in Theorem IX.3.1.

Proof. (existence of $r, d_1, \ldots, d_m$ in Theorem IX.3.1.) Let A be a finitely generated abelian group. By Theorem IX.1.3 we have $A \cong \mathbb{Z}^n/H$ for some subgroup $H \leqslant \mathbb{Z}^n$. Choose bases $f_1, \ldots, f_n$ of $\mathbb{Z}^n$ and $d_1 f_1, \ldots, d_k f_k$ of H as described in Theorem IX.3.4. Under the isomorphism $\mathbb{Z}^n \cong \mathbb{Z}^n$ given by $\sum a_i f_i \mapsto (a_1, \ldots, a_n)$ the subgroup H is mapped to the subgroup $d_1\mathbb{Z} \times d_2\mathbb{Z} \times \ldots \times d_k\mathbb{Z} \times (0) \times \ldots \times (0)$. Now consider

$$\varphi : \mathbb{Z}^n \longrightarrow \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_k\mathbb{Z} \times \mathbb{Z}^{n-k}$$

given by $\varphi(a_1, \ldots, a_n) = (a_1 \bmod d_1, \ldots, a_k \bmod d_k, a_{k+1}, \ldots, a_n)$. This is a surjective homomorphism with kernel $d_1\mathbb{Z} \times d_2\mathbb{Z} \times \ldots \times d_k\mathbb{Z} \times (0) \times \ldots \times (0)$. Hence Theorem VIII.2.1 shows that

$$\begin{aligned} A \cong \mathbb{Z}^n/H &\cong \mathbb{Z}^n/(d_1\mathbb{Z} \times d_2\mathbb{Z} \times \ldots \times d_k\mathbb{Z} \times (0) \times \ldots \times (0)) \\ &\cong \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_k\mathbb{Z} \times \mathbb{Z}^{n-k}. \end{aligned}$$

Removing the factors $\mathbb{Z}/1\mathbb{Z} \cong (0)$ from this product shows the existence stated in Theorem IX.3.1. ∎

It remains to prove that the integers $r, d_1, \ldots, d_m$ in Theorem IX.3.1 are unique. To this end, the following notion will be crucial:

IX.3.6 Definition. Let A be an abelian group. The set $A_{\text{tor}} = \{a \in A \mid \text{ord}(a) < \infty\}$ is a subgroup of A called the *torsion subgroup* of A.

Verify yourself that indeed A_{tor} is a subgroup and that the only element of finite order in the factor group A/A_{tor} is its its unit element A_{tor} If A is finitely generated, then we know $A \cong \mathbb{Z}^r \times \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_m\mathbb{Z}$ and here the second group has as elements of finite order precisely the elements $(0, \ldots, 0, \overline{a_1}, \ldots, \overline{a_m})$, for $\overline{a_i} \in \mathbb{Z}/d_i\mathbb{Z}$. Therefore we get $A_{\text{tor}} \cong \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_m\mathbb{Z}$. Moreover, $A/A_{\text{tor}} \cong \mathbb{Z}^r$: indeed, consider the composition

$$A \cong \mathbb{Z}^r \times \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_m\mathbb{Z} \longrightarrow \mathbb{Z}^r$$

where the second map is projection onto the first r coordinates. This is a surjective homomorphism with kernel A_{tor}. Hence Theorem VIII.2.1 implies $A/A_{\text{tor}} \cong \mathbb{Z}^r$.

This discussion implies in particular that for a finitely generated abelian group A, the integer r in Theorem IX.3.1 equals the rank of the finitely generated free group A/A_{tor}. Hence by Theorem IX.2.4 it is unique.

We moreover conclude that $d_1 \cdot \ldots \cdot d_m = \#(\mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_m\mathbb{Z}) = \#A_{\text{tor}}$ and therefore the product of the integers d_1 up to d_m does not depend on the actual choice of integers as in Theorem IX.3.1.

Arguments like this will imply the uniqueness $d_1, \ldots, d_m$. We begin by showing some more properties of these integers. The number d_1 (this is the largest elementary divisor) is unique. Namely d_1 is the largest integer appearing as the order of an element in $\mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_m\mathbb{Z}$ (check this!). This equals the maximal order of elements in A_{tor} which determines it.

Also the *number* of elementary divisors is fully determined by A. Namely, take a prime number p. Multiplying by p defines a homomorphism $A \to A$. Its kernel we write as $A[p]$. This is a subgroup of A and of A_{tor}. The number of elements in $A[p]$ equals the number of elements $(\overline{a_1}, \ldots, \overline{a_m}) \in \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_m\mathbb{Z}$

with $p(\overline{a_1},\ldots,\overline{a_m}) = (\overline{0},\ldots,\overline{0})$. A small calculation shows that this equals p^k, with k the number of indices i such that $p|d_i$. This k is maximal when $p \mid d_m$, in which case $k = m$. So we conclude that m equals the maximal exponent k such that a prime p exists with $\#A[p] = p^k$. This determines m in terms of A.

Proof. (of the uniqueness of the elementary divisors in Theorem IX.3.1.) We show the uniqueness of $d_1,\ldots,d_m$ by mathematical induction with respect to the positive integer $\#A_{\text{tor}} = d_1 \cdot \ldots \cdot d_m$. For $\#A_{\text{tor}} = 1$ the sequence of elementary divisors is empty so in this case uniqueness holds. Suppose $\#A_{\text{tor}} = N > 1$ and assume the uniqueness for all A' with $\#A'_{\text{tor}} < N$. Let $d_1,\ldots,d_m$ be a sequence of elementary divisors for A. We have $d_m > 1$ since $N > 1$. Take a prime $p|d_m$ and consider the factor group $A' = A/A[p]$. Since A is finitely generated, so is A' and $A'_{\text{tor}} \cong \mathbb{Z}/\frac{d_1}{p}\mathbb{Z} \times \ldots \times \mathbb{Z}/\frac{d_m}{p}\mathbb{Z}$. The induction hypothesis implies that the sequence $d_1/p,\ldots,d_m/p$ is unique, which implies the same is true for d_1 up to d_m. ∎

We finally discuss subgroups of $\mathbb{Z}^n$ generated by n elements. In particular we will decide when such a subgroup has finite index in $\mathbb{Z}^n$.

IX.3.7 Theorem. *Suppose that $H \leqslant \mathbb{Z}^n$ is a subgroup generated by n elements $g_1,\ldots,g_n$ and $g_i = a_{1i}e_1 + \ldots + a_{ni}e_n$ for some basis $\{e_1,\ldots,e_n\}$ of $\mathbb{Z}^n$. Let $A = (a_{ij})$ be the corresponding $n \times n$ matrix. Then H has finite index in $\mathbb{Z}^n$ if and only if* $\det(A) \neq 0$. *If* $\det(A) \neq 0$ *holds, then* $\#\mathbb{Z}^n/H = [\mathbb{Z}^n : H] = |\det(A)|$.

Proof. By the method described in the proof of Theorem IX.3.4 one transforms A into a diagonal matrix with nonnegative integers $d_1,\ldots,d_n$ on the diagonal. Note that the steps in this procedure can only change the sign of the determinant. In particular $|\det(A)| = d_1 \cdot \ldots \cdot d_n$. The results of the present section show that $\mathbb{Z}^n/H \cong \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_n\mathbb{Z}$. The latter group is finite precisely when all d_i's are different from 0. If this is the case then $[\mathbb{Z}^n : H] = \#\mathbb{Z}^n/H = d_1 \cdot \ldots \cdot d_n = |\det(A)|$. ∎

IX.3.8 Example. Take $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 2 \\ 3 & 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 3 & 4 & 2 \end{pmatrix}$.

One computes $\det(A) = 0$ and $\det(B) = 4$. For the subgroups $H_1 = A \cdot \mathbb{Z}^3$ and $H_2 = B \cdot \mathbb{Z}^3$ of $\mathbb{Z}^3$ we therefore conclude that the factor group $\mathbb{Z}^3/H_1$ is infinite and $\mathbb{Z}^3/H_2$ consists of 4 elements. The method described in the proof of Theorem IX.3.4 transforms A into a diagonal matrix with entries $1,2,0$ on the diagonal. Hence $\mathbb{Z}^3/H_1 \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Similarly B is transformed into the diagonal matrix with entries $1,2,2$. So $\mathbb{Z}^3/H_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The element of order 2 in $\mathbb{Z}^3/H_1$ is the class $(0,1,1) + H_1$. Namely the given element is not the zero element in $\mathbb{Z}^3/H_1$ because this would imply $(0,1,1) \in H_1$. Since the second coordinate of any element in H_1 is even, this is not the case. The order of $(0,1,1) + H_1$ is indeed 2 since $2 \cdot ((0,1,1) + H_1) = (0,2,2) + H_1 = (0,0,0) + H_1$, as $(0,2,2) \in H_1$.

Analogously, try to find the three distinct elements of order 2 in $\mathbb{Z}^3/H_2$! ∎

IX.4 Exercises

1. Show that the multiplicative group $(\mathbb{Q}\setminus\{0\},\cdot,1)$ is not finitely generated.
2. Show that if N is a normal subgroup of a finitely generated group G, then the factor group G/N is finitely generated as well.
3. Consider the group $\mathrm{SL}_2(\mathbb{Z})$ of all matrices in $\mathrm{SL}_2(\mathbb{Z})$ with integral entries and determinant 1. Let $S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right), T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$. Compute $U=ST$ and show that $\mathrm{ord}(S)=4$ and $\mathrm{ord}(U)=6$ and S and that U generate the group $\mathrm{SL}_2(\mathbb{Z})$.
4. Write the matrix $\left(\begin{smallmatrix}55&21\\34&13\end{smallmatrix}\right)$ as a product of powers of $S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.
5. Consider the group $\mathrm{GL}_2(\mathbb{Z})$ of all matrices in $\mathrm{GL}_2(\mathbb{R})$ with integral entries and determinant ±1. Show that $\mathrm{GL}_2(\mathbb{Z})$ is finitely generated and give explicit generators all having finite order (for instance three generators suffice, of order $2,4$, and 6, respectively).
6. Present an alternative proof for the fact that $\mathbb{Z}^{k_1}\not\cong\mathbb{Z}^{k_2}$ in case $k_1\neq k_2$, by verifying (and using) that a basis for $\mathbb{Z}^k$ is in fact also a basis for the vector space $\mathbb{R}^k$ over $\mathbb{R}$.
7. Find a basis for the subgroup $H=\{(a,b,c,d)\mid a+b+c+d=0 \text{ and } a\equiv c \bmod 12\}$ of $\mathbb{Z}^4$.
8. Determine the rank and the elementary divisors of each of the following groups.
   (a) $\mathbb{Z}\times 17\mathbb{Z}\times\mathbb{Z}/18\mathbb{Z}\times\mathbb{Z}/12\mathbb{Z}$.
   (b) $(\mathbb{Z}/15\mathbb{Z})^\times$
   (c) $(\mathbb{Z}/17\mathbb{Z})^\times$
   (d) $\mathbb{Z}^3$ modulo the subgroup generated by $(1,2,0)$ and $(3,0,0)$.
   (e) A/H with $A\leqslant\mathbb{Z}^5$ the group of all 5-tuples with sum 0 and $H=A\cap B(\mathbb{Z}^5)$ where $B=\begin{pmatrix}-13&1&1&0&0\\1&-13&1&0&0\\1&1&-1&1&1\\0&0&1&-2&0\\0&0&1&0&-3\end{pmatrix}$.
9. Find the number of pairwise non-isomorphic abelian groups consisting of 72 elements.
10. (a) Use that $5^{2^{n+1}}-1=(5^{2^n}-1)(5^{2^n}+1)$ to prove that $5^{2^n}-1$ contains exactly $n+2$ factors 2.
    (b) Conclude from (a) that the order of $\overline{5}$ in $(\mathbb{Z}/2^n\mathbb{Z})^\times$ equals 2^{n-2} (for $n\geq2$).
    (c) Show that for $n\geq2$ the map $a \bmod 2^n\mapsto a \bmod 4$ is a well defined surjective homomorphism from $(\mathbb{Z}/2^n\mathbb{Z})^\times$ to $(\mathbb{Z}/4\mathbb{Z})^\times$, and its kernel is the subgroup generated by $\overline{5}$.
    (d) Determine the number of elements of order ≤2 in $(\mathbb{Z}/2^n\mathbb{Z})^\times$, and use this to find the rank and the elementary divisors of $(\mathbb{Z}/2^n\mathbb{Z})^\times$.
11. (a) Prove that if A is a finite abelian group and p is a prime with $p\nmid\#A$, then $A/pA\cong(0)$.
    (b) Show that for $A=\mathbb{Z}/N\mathbb{Z}$ and p a prime with $p|N$ one has $A/pA\cong\mathbb{Z}/p\mathbb{Z}$.
    (c) Prove that if A is a finitely generated abelian group and p is a prime, then $\#A/pA=p^k$ where k equals the sum of the rank of A and the number of indices i such that the elementary divisor d_i of A is divisible by p.
12. Let $d\geq3$ be an integer. In this problem we study the polynomial X^2+X+d. Let $\alpha_d\in\mathbb{C}$ be a zero of this polynomial and define $A_d=\{a+b\alpha_d\mid a,b\in\mathbb{Z}\}\subset\mathbb{C}$.
    (a) Show that A_d is a subgroup of the additive group $(\mathbb{C},+,0)$ and $A_d\cong\mathbb{Z}^2$.
    (b) Take $\beta=a+b\alpha_d\in A_d$. Show that $\beta A_d=\{\beta\cdot\gamma\mid\gamma\in A_d\}$ is a subgroup of A_d, and if $\beta\neq0$ then $\#A_d/\beta A_d=a^2-ab+db^2$.

(c) Let a be an integer satisfying $0 \le a \le d-2$ such that a^2+a+d is not prime. Let p be the smalles prime dividing a^2+a+d. Prove that $p \le d-1$.

(d) Given a and p as above, let $H = pA_d+(a-\alpha_d)A_d = \{p\gamma+(a-\alpha_d)\delta \mid \gamma,\delta \in A_d\}$. Show that $H \subset A_d$ is a subgroup generated by p and $p\alpha_d, a-\alpha_d, d+(a+1)\alpha$. Conclude that $\#A_d/H = p$.

(e) Use (b) and (d) to conclude that H is not of the form βA_d, for any $\beta \in A_d$. Show that H is closed under multiplication by elements of A_d; this means that for every $h \in H$ and every $\gamma \in A_d$ the product $h\gamma$ is an element of H.

(f) Conclude that if an integer a exists with $0 \le a \le d-2$ and a^2+a+d *not* a prime number, then A_d contains a subgroup which is closed under multiplication by elements of A_d, and this subgroup cannot be written as βA_d for any $\beta \in A_d$.

(g) It is a fact from "*algebraic number theory*" that A_{41} and A_{17} have the property that all subgroups closed under multiplication by all elements of A_d have the form βA_d. Draw a conclusion from this concerning the polynomials X^2+X+17 and X^2+X+41.