Analyzing Network Dynamics through Graph Partitioning

Shuo Zhang

The research for this doctoral dissertation has been carried out at the Faculty of Mathematics and Natural Sciences, University of Groningen, the Netherlands, within a collaboration between the *Research Institute of Industrial Engineering and Management* and the *Johann Bernoulli Institute for Mathematics and Computer Science.*

The research reported in this dissertation is part of the research program of the Dutch Institute of Systems and Control (DISC). The author has successfully completed the educational program of the Graduate School DISC.

Printed by IPSKAMP Drukkers B.V.
Enschede, the Netherlands

rijksuniversiteit
groningen

Analyzing Network Dynamics through Graph Partitioning

Proefschrift

ter verkrijging van de graad van doctor aan de
Rijksuniversiteit Groningen
op gezag van de
rector magnificus, prof. dr. E. Sterken
en volgens besluit van het College voor Promoties.

De openbare verdediging zal plaatsvinden op

vrijdag 21 februari 2014 omm 11 uur

door

Shuo Zhang

geboren op 9 december 1981
te Changchun, China

Promotors
Prof. dr. M. K. Camlibel
Prof. dr. M. Cao
Prof. dr. ir. J. M. A. Scherpen

Beoordelingscommissie
Prof. dr. J. Lunze
Prof. dr. C. Hadjicostis
Prof. dr. H. L. Trentelman

ISBN (Book): 978-90-367-6804-7
ISBN (Ebook): 978-90-367-6803-0

Contents

Acknowledgments

It is never simple to attribute a Ph.D thesis to a single person or even a certain group, especially when mine is an output of a four-year experience studying abroad.

I first give thanks to my promotors. My professional skills are positively affected by Prof. dr. M. Kanat Camlibel. In a short term, these skills are built up on the knowledge inherited from Kanat of linear algebra and the geometric control approach, which in turn produces this thesis. In a long term, I am gradually influenced by Kanat's respectful attitude towards scientific research, including his strong passion in investigating problems, his tendancy to make every statement to be precise as well as his concise way of mastering complex concepts. Prof. dr. Ming Cao strongly supported me to win the Ubbo Emmius scholarship, without which I would have not an opportunity to do my research. I also thank Ming for introducing me to the topic of my thesis. As my team leader, Prof. dr. ir. Jacquelien Scherpen coordinates the team efficiently so that I can work in a memorable environment.

Four years Ph.D life does not only mean to prove mathematical theorems. To study abroad does not simply mean I only need to switch my working language to English or to learn some Dutch. I also need friends in my daily life and fortunately I have. Gunn Larsen and Ewoud Vos are such nice friends (and also my defense paranymphs) that they often offer help even without being requested. Ewoud helps to translate my summary to Dutch. Gunn contributes to my cover design. Nima Monshizadeh often has coffee breaks with me (and also with Gunn and Ewoud). His optimism towards life also makes me to keep positive. Moreover, Nima and I coauthor two papers and I get much benefit from him. Hector Garcia de Marina Peinado, Mauricio Munos Arias and James Riehl are also good friends to discuss technical problems and to share daily life issues. As the secretary of the team, Frederika Fokkens usually provides help with a warm heart. I thank all the aforementioned friends and others whose names cannot be listed here due to the page limit (a pity indeed!). I take this chance to say: thank you all for giving your souls to me.

I specially give my thanks to my parents. I know how lonely you feel when your unique son is so far away from you for four years. You always keep me in your hearts and feel proud of your son. Your love gives me the confidence of overcoming all the challenges in my life! Also thanks to you for painting traditional Chinese pictures for my thesis cover.

I also appreciate the effort of the reading committee on improving the quality of my thesis.

As a final word, the Chinese symbol indicating "human beings" has a structure to support each other. The philosophy hidden behind this symbol becomes apparent to me when I write this thesis for my degree of Doctor of Philosophy.

Shuo Zhang
Groningen
January 19, 2014

Chapter 1

Introduction

In this thesis, we refer to a multi-agent network as a group of dynamical agents that are coupled together by exchanging local information between them. One source for inspiration of the study of networks is the reported collective behaviors of biological groups in nature. For instance, a flock of migrating birds form the well-known V-shape that is advantageous to reduce the drag force on each individual bird while guaranteeing sufficient visibility between them. Moreover, the V-shape is formed by the bird group under the constraint that each individual bird only perceives its local surrounding environment and that there is hardly any command from a central coordinator. In fact, one of the most interesting phenomena in a network is similar to the above observation: although each agent in a network has access to only *local* information, some desired *global* behavior may surprisingly emerge. Network scientists have made a concerted effort to investigate the mechanism behind the above intriguing phenomena. To do so, they have proposed protocols and studied the simulated behaviors of networks under such protocols. For instance, a protocol based on a nearest neighbor rule has been employed for a group of self-propelled particles, see e.g. [54]. As a result, all the particles move in the same direction by knowing only the heading of their neighbors, although none of them follows a global coordination rule and each of them is under the influence of noise.

From an engineering perspective, the ultimate goal of understanding mechanisms behind desired network behaviors is to design controllers for networks such that the networks have some desired properties. Unfortunately, the tools in classical control theory are not readily applicable to some network problems for a couple of reasons. Firstly, network models are usually high dimensional. It is inconvenient or even impractical to apply some well-established techniques such as optimal control strategies without any modification to such large-scale systems. Further, controllers synthesized by classical control theory usually do not take into account explicitly the network topologies that arise from the information exchange between agents. The design of a controller can be simplified and more practical if it can utilize the network topology information as well as the agents' sensed local information.

To come up with effective network control methods, one can view network synthesis problems from a system theoretical perspective. One pioneering paper in this research direction is [24]. By modeling the behavior of the network with the nearest neighbor rule in [54] as a switched linear system, the paper [24] shows that whether all the particles can move in the same direction is equivalent to whether a switched linear system is stable. The network with the protocol based on the nearest neighbor rule in [24] is in fact what we call a *diffusively coupled network* in this thesis. The results in [24] have attracted growing interest in studying fascinating behaviors of diffusively coupled networks from a system theoretical point of view such as consensus [24], synchronization [1, 47] and clustering synchronization [30, 57, 58]. The observed phenomenon in [54] is in fact the consensus of the self-propelled particle network on their moving direction. In general, a network is said to achieve consensus if the states of all its agents finally agree on a common value, which is usually dependent on the initial states of all the agents. As shown in [24], whether a network achieves consensus is equivalent to whether some corresponding dynamical system is output stable.

The study of system theoretical properties of diffusively coupled networks further breaks new ground for the study at the interface of control theory and graph theory, see e.g. [14]. Indeed, the network topology determined by diffusive couplings can always be described by a *graph*: each vertex of the graph represents each agent of the network and each edge indicates the information exchange between a pair of agents. The results in [24] show that whether a network can reach consensus depends on the connectivity of its underlying graph. Providing graph theoretical conditions for system theoretical properties leads to distributed control strategies which control networks to achieve desired behaviors. One class of desired behaviors are to specify, achieve and maintain formations for a team of autonomous agents. Various formation control strategies have been studied, see e.g. [6, 7, 18, 39].

Although we have witnessed encouraging progress on distributed control of diffusively coupled networks, there are still fundamental issues that have not been fully addressed. For instance, one intuitive idea to control a network is as follows: a direct control action is applied to some agents and is propagated to the other agents of the network through the couplings. The agents directly receiving control inputs are usually called *leaders* and the others are called *followers*. This idea has been successfully employed to develop *pinning control* methods, see e.g. [48]. Pinning control methods are usually taken for networks to achieve synchronization by manipulating leaders with given control laws. In order to synthesize a network under the leader/follower framework, it is naturally interesting to know which agents should be chosen to be the leaders such that all the other agents can be affected in a desirable way directly or indirectly. This question can be answered by studying *controllability* of networks. Intuitively, a network is said to be “controllable” if, by manipulating its leaders, the overall network can be made to behave in some desirable way. Roughly speaking, there are three related concepts that have been made precise for networks. Two of them are the so-called *weak structural controllability* [29] and *strong structural controllability* [17], respectively. In studying weak/strong structural controllability of networks, one is allowed to give weights to the couplings, which indicate the strengths of couplings between agents. However, one is not allowed to add new couplings, which is often expensive. If the network is "controllable" by manipulating the leaders for *at least one* set of weights, then the network is said to be weakly structurally controllable. In case the network is "controllable" for *any* possible set of weights, then it is defined to be strongly structurally controllable. The results in [17, 29] reveal that weak/strong structural controllability of diffusively coupled networks in fact have clear graph theoretical characterizations in terms of the so-called *maximum/constrained matchings*, which in turn give indications on how to choose as few leaders as possible. Another notion is the standard concept of *controllability* in control theory for the network. To study whether a network is controllable, the weights of the couplings are given a priori. Although the weak/strong structural controllability of a network already has clear graph theoretical characterizations, the graph theoretical characterization of controllability of networks is still an open question, see e.g. [13, 31, 51]. As pointed out in [51], "the lack of a graph theoretical characterization of the controllability property prevents us from building controllable interconnection topologies" and cannot lead to network synthesis such as choosing leaders (as few as possible) to guarantee the network to be controllable.

In this thesis, we mainly provide graph theoretical characterizations for three system theoretical properties of diffusively coupled networks, namely *controllability*, *partial consensus* and *disturbance decoupling*. Our study on controllability mainly follows and extends the existing results in [31]. To provide graph theoretical conditions for controllability, we take the *geometric approach* in linear control theory, see e.g. [2, 52, 55]. In particular, we set diffusively coupled networks in linear state spaces and provide graph theoretical characterizations for *controllable subspaces*. At the same time, we also address two other significant issues. One of them aims at revealing the roles that agent dynamics and the network topology play in network controllability. There have been results along this direction for higher-order-integrator agents, see e.g. [22, 25]. However, we will focus on networks with general linear dynamical agents. The other issue is a synthesis problem: how to choose leaders to guarantee the network to be controllable. There have been existing results in this research direction, see e.g. [5, 10, 35, 36, 40]. Instead, we focus on networks with distance regular topologies and provide systematic ways of choosing agents to guarantee network controllability. Our graph theoretical characterizations of controllability of networks can be immediately extended to the scenarios where network topologies are switching in time. The study on networks with switching topologies is motivated by various applications such as power networks, see e.g. [24, 33, 37, 46].

The partial consensus problem studies networks with double-integrator (position/velocity) agents. Here, the word "partial" is used to emphasize the scenarios where not necessarily all the positions and/or the velocities finally achieve some common values. Compared to existing results on consensus of double-integrator networks such as [43, 44, 61, 62], the topologies arising from position information exchange and from velocity information exchange are different. The partial consensus problem of heterogeneous networks with double-integrator agents has been proposed by [21], where algebraic conditions for partial consensus have been provided. We are more interested in providing graph theoretical conditions after further developing algebraic conditions.

The disturbance decoupling problem (DDP) of a diffusively coupled network studies how to decouple some output of interest of the network from the effect of external disturbances. The DDP of a linear system has been well studied in linear control theory by the geometric approach, see e.g. [3, 52, 55, 56]. However, DDP has not been studied for diffusively coupled networks.

1.1 Outline of this thesis

In Chapter 2, we study controllability of diffusively coupled networks with time-independent topologies. We first reveal the effect of general linear (agent) dynamics and of the network topology on overall network controllability. After commenting on existing graph theoretical conditions, we mainly provide both the lower and the upper bounds for the controllable subspace of the network by two classes of graph partitions called distance partitions and almost equitable partitions. We further illustrate that these bounds cannot be improved in terms of the characteristic matrices of graph partitions. Finally, we provide a systematic way of choosing agents such that networks with distance regular topologies can be controlled by manipulating these agents. Chapter 2 is based on our papers [13, 15, 63, 64].

In Chapter 3, we provide graph theoretical conditions for controllability of diffusively coupled networks with switching topologies. The conditions in Chapter 3 can be extended from Chapter 2 immediately. Chapter 3 is based on our paper [65].

In Chapter 4, we study the *partial* consensus problem for heterogeneous networks with double-integrator (position/velocity) agents. From a system theoretical perspective, we first formulate the partial consensus problem to be an output stability problem. Then we provide both the algebraic and the graph theoretical conditions. The graph theoretical conditions we develop are in terms of a class of graph partitions that can be taken as a subclass of almost equitable partitions in Chapter 2. The results in Chapter 4 are from our published paper [12].

In Chapter 5, we aim at decoupling some given output of the network from external disturbances injected into the network. In classical linear systems, the solution of disturbance decoupling problem (DDP) is given in terms of *controlled invariant subspaces*. To provide graph theoretical conditions to solve the DDP for networks, we introduce and develop a class of generalized almost equitable partitions that induce a class of controlled invariant subspaces in networks. Moreover, we see how network synthesis benefits from graph theoretical conditions: the solution of DDP (if it exists) *only* requires the relative information of network agents, which is a more realistic solution in networks. The presentation in Chapter 5 is based on our submitted paper [34].

In Chapter 6, we reemphasize the common thread of this thesis and summarize our contributions. We also mention some topics that may become interesting in the future.

Chapter 2

Controllability of diffusively coupled networks

2.1 Introduction

Recently, significant work has been done to study distributed and cooperative control of multi-agent networks [9, 26]. It is of particular interest to study the case when the agents are coupled together through linear diffusive couplings since rich collective behaviors, such as synchronization [1] and clustering [30, 58], may arise as a result of local interactions among agents without centralized coordination or control. To reduce the complexity of controller design, one is especially interested in knowing how to influence the behavior of the overall network by just controlling a small fraction of the agents [48, 57]. We call such agents that are under the forcing of external control inputs the *leaders* and correspondingly the rest of the agents *followers*. Hence, to study whether any desired collective behavior can be achieved in finite time by controlling the leaders is equivalent to the study of the controllability of the overall networks consisting of all the leaders and followers. For example, the controllability problem has been related to the problem of controlling a formation of mobile robots by manipulating the trajectories of the leaders such that all the robots can move from any initial positions to any desired final positions within finite time [32].

The controllability of diffusively coupled multi-agent networks was first studied in [51] and later in [5, 15, 19, 28, 31, 40, 41, 59]. However, most existing results deal with networks where agents have single-integrator dynamics, except agents with double-integrators [22] and agents with higher-order integrators [25]. In this chapter, we first study diffusively coupled networks where agents have identical general linear dynamics. We reveal in Theorem 2.2.1 the dependence of controllability of such networks on agent dynamics and network topologies. Existing results on agents with higher-order-integrator dynamics [22, 25] can be considered as special cases of Theorem 2.2.1.

Later, we focus on inferring network controllability from its topology. To do this, we consider diffusively coupled networks with single-integrator agents. We comment on existing results [31, Prop. 2] and [31, Thm. 3] and provide counterexamples to show that these results are erroneous.

For diffusively coupled networks with single-integrator agents, we provide

i. a dimensional lower bound in terms of the distance partitions (Theorem 2.4.2), and
ii. an upper bound in terms of the maximal almost equitable partitions (Theorem 2.4.8)

for their controllable subspaces.

The contribution of these results is two-fold. Firstly, the distance partitions in Theorem 2.4.2 yield easily computable lower bounds. Secondly, the upper bounds we provide are valid in the case of multi-leader scenarios unlike the existing upper bounds in the literature [15, 31] which deal with *only* single-leader cases. Also, we provide an algorithm for obtaining the maximal almost equitable partition for given leaders.

The bounds mentioned above are developed without imposing any extra structure on the underlying graph topologies. When the underlying graph has the so-called distance regularity property, we show that the controllable subspace can be fully characterized if there is a single leader. In the case of multiple leaders, we present a necessary and a sufficient condition for controllability. Finally, we discuss how to choose leaders among the agents in order to guarantee controllability when the graphs have cycles and complete topologies.

This chapter is organized as follows. In Section 2.2, we first introduce the class of diffusively coupled networks that is of our interest and later reveal the effect of agent dynamics and network topologies on network controllability. Before our study on inferring controllability from topologies, we provide counterexamples in Section 2.3 to comment on existing results in [31, Prop. 2] and [31, Thm. 3] that are in the same research line. Section 2.4 is devoted to the study of the effect of the underlying topology on the controllability. In this section, we provide a lower bound for the controllable subspace of such a network in terms of distance partitions and an upper bound in terms of the so-called almost equitable partitions. Section 2.4 also presents an algorithm in order to compute the almost equitable partition which bounds the controllable subspace from above. Following Section 2.4, we focus on networks with distance regular topologies in Section 2.5.

2.2 Networks and controllability

2.2.1 Diffusively coupled multi-agent networks

Consider a diffusively coupled network consisting of n agents labeled by the set $\mathcal{V} = \{1, 2, \dots, n\}$ where n is a positive integer. We assign the roles of leaders and followers to the agents and define $\mathcal{V}_{\mathrm{L}} = \{v_1, v_2 \ldots, v_m\}$ where m is a positive integer with $m \leqslant n$ and $\mathcal{V}_{\mathrm{F}} = \mathcal{V} \setminus \mathcal{V}_{\mathrm{L}}$ to denote the sets of indices of the leaders and followers, respectively.

To each follower $i \in \mathcal{V}_{\mathrm{F}}$, we associate a linear dynamical system

$$\dot{x}_i = Ax_i + Cz_i$$

and to each leader $i \in \mathcal{V}_{\mathrm{L}}$ with $i = v_\ell$ a linear dynamical system

$$\dot{x}_i = Ax_i + Cz_i + Bu_\ell$$

where $x_i \in \mathbb{R}^p$ denotes the state of agent i, $u_\ell \in \mathbb{R}^q$ the external input to agent $i = v_\ell$, $z_i \in \mathbb{R}^s$ the coupling variable for the agent i, and all matrices involved are of appropriate dimensions.

Two distinct agents i and j are said to be *neighbors* if their states are known by each other. Throughout this chapter, we assume that the neighboring relationships are time-independent. Such neighboring relationships can be described by a simple undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the vertex set and $\mathcal{E}$ is the edge set such that $(i, j) \in \mathcal{E}$ if agents i and j are neighbors. The coupling variable z_i for each agent $i \in \mathcal{V}$ is determined by the so-called diffusive coupling rule based on the neighboring relations as follows:

$$z_i = K \sum_{(i,j)\in\mathcal{E}} (x_j - x_i)$$

where $K \in \mathbb{R}^{s\times p}$ is the matrix describing the coupling strengths.

By defining $x = \operatorname{col}(x_1, x_2, \ldots, x_n)$ and $u = \operatorname{col}(u_1, u_2, \ldots, u_m)$, we can write the above leader-follower linearly diffusively coupled multi-agent network associated with the graph $\mathbb{G}$ into a compact form as

$$\dot{x} = \hat{L}x + \hat{M}u \tag{2.1}$$

where $\hat{L} = I_n \otimes A - L \otimes CK$ with L being the Laplacian matrix of the graph G, $\hat{M} = M \otimes B$ with $M \in \mathbb{R}^{n\times m}$ defined by

$$M_{i\ell} = \begin{cases} 1 & \text{if } i = v_\ell \\ 0 & \text{otherwise.} \end{cases}$$

Here "$\otimes$" denotes the Kronecker product [4]. Note that the matrix product CK is sometimes called the *inner coupling matrix* in the study of the synchronization behavior in complex networks [1]. In the next subsection, we study the controllability of the network (2.1).

2.2.2 Effects of agent dynamics and network topology on controllability

In the following theorem, we provide necessary and sufficient conditions for the controllability of the network (2.1).

Theorem 2.2.1 *The pair $(\hat{L}, \hat{M})$ is controllable if and only if the pair (L, M) is controllable and for each eigenvalue λ of L, the pair $(A - \lambda CK, B)$ is controllable.*

Proof. (Necessity) We only prove the necessity of the controllability of the pair (L, M), and the necessity of the controllability of the pair $(A - \lambda CK, B)$ can be proved in a similar manner. Suppose (L, M) is uncontrollable. Then there exists some nonzero $x \in \mathbb{R}^n$ such that $x^T L = \lambda x^T$ and $x^T M = 0$. Let $(\theta, y) \in \mathbb{C} \times \mathbb{C}^p$ be a left-eigenpair of the matrix $A - \lambda CK$. Note that $(\theta, x \otimes y)$ is a left-eigenpair of $\hat{L}$ and $(x \otimes y)^H \hat{M} = (x^T M) \otimes (y^H B) = 0$ where z^H denotes the conguate transpose of the vector z. This implies that the pair $(\hat{L}, \hat{M})$ is uncontrollable.

(Sufficiency) On the contrary, suppose that $(\hat{L}, \hat{M})$ is uncontrollable. Since L is symmetric, one can always find an orthonormal matrix U such that $U^T L U = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_i's are eigenvalues of L. Now consider the following two matrices $\tilde{L}$ and $\tilde{M}$ that are obtained from $\hat{L}$ and $\hat{M}$ respectively according to

$$\tilde{L} \triangleq (U^T \otimes I_p)\hat{L}(U \otimes I_p) = \mathrm{blockdiag}(A - \lambda_1 CK, \ldots, A - \lambda_n CK)$$

and

$$\tilde{M} \triangleq (U^T \otimes I_p)\hat{M} = (U^T \otimes I_p)\hat{M} = U^T M \otimes B.$$

Since $(\hat{L}, \hat{M})$ is uncontrollable and $U^T \otimes I_p$ is nonsingular, the pair $(\tilde{L}, \tilde{M})$ is also uncontrollable. In view of the block diagonal structure of $\tilde{L}$, we know that there must exist an index s with $1 \leqslant s \leqslant n$ such that the corresponding matrix pair $(A - \lambda_s CK, (U^T M)_s \otimes B)$ is uncontrollable, where for a matrix M we use $(M)_s$ to denote its sth row. This, however, implies that (L, M) is uncontrollable in case $(U^T M)_s = 0$ or $(A - \lambda_s CK, B)$ is uncontrollable in case $(U^T M)_s \neq 0$. Hence, we arrive at a contradiction. ■

The main results of [22, 25] on the controllability of networks of agents with higher-order-integrator dynamics can be recovered from Theorem 2.2.1 as special cases. In fact, the networks in [22, 25] correspond to a particular case of (2.1) by setting

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad C = B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} c_1 & c_2 & \cdots & c_{p-1} & c_p \end{bmatrix}$$

where $A \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^p$, $C \in \mathbb{R}^p$, and $K^T \in \mathbb{R}^p$ with c_i's being the coupling strengths for $1 \leqslant i \leqslant p$. From Theorem 2.2.1, one immediately concludes that the network pair $(\hat{L}, \hat{M})$ is controllable if and only if (L, M) is controllable. This is coincide with the results in [22, 25].

Theorem 2.2.1 also reduces the computational cost for checking controllability because the dimensions of the controllability matrices of the pairs (L, M) and $(A - \lambda CK, B)$ are much lower than that of the pair $(\hat{L}, \hat{M})$ when the number of agents n is large or the dimension p of agent dynamics is high.

One can roughly interpret the two conditions stated in Theorem 2.2.1 to be the effects of network topologies and agent dynamics on controllability. Next, we are especially interested in inferring network controllability from its underlying topology. To do this, we consider diffusively coupled networks with single-integrator agents as follows:

$$\dot{x} = -Lx + Mu. \tag{2.2}$$

Obviously, such single-integrator networks are special cases of the network (2.1) by setting $A = 0$ and $B = C = K = 1$.

In the next section, we comment on existing results that characterizes network controllability in terms of underlying graph topologies.

2.3 A class of uncontrollable networks

The paper [31] deals with diffusively coupled networks with single-integrator agents and one of the agents, the leader, provides the control input. In order to study the controllability problem for such networks, it introduces the so-called leader-invariant relaxed equitable partition (LEP) of the underlying graph [31, Def. 5]. The LEP is then employed in [31, Thm. 3] and in [31, Prop. 2]. The former result states that a single leader network is controllable if and only if the LEP of the graph is the trivial partition whereas the latter characterizes the controllable subspace of a single leader leader-symmetric network [31, Def. 1] in terms of the characteristic vectors obtained from the LEP. In this section, we provide counterexamples for these two results.

The framework that [31] considers for controllability is different than ours in (2.2). The following subsection shows that they are in fact equivalent for studying controllability.

2.3.1 Two frameworks for studying controllability

In this subsection, we relabel the agents such that the first $n-m$ agents are followers and the last m agents are leaders. Then one can partition L into block submatrices

$$L = \begin{bmatrix} L_f & l_{fl} \\ l_{fl}^T & L_l \end{bmatrix}$$

where L_f and L_l are $(n-m)\times(n-m)$ and $m\times m$ dimensional matrices respectively. Let $x_l \in \mathbb{R}^m$ denote the state of the leaders and $x_f \in \mathbb{R}^{n-m}$ that of the followers. Then in [31] the following model is studied

$$\dot{x}_f = -L_f x_f + l_{fl} x_l \tag{2.3}$$

where x_l is taken as the control input.

Now we compare the controllable subspaces $\mathcal{K}(L, M)$ of (2.2) and $\mathcal{K}(L_f, l_{fl})$ of (2.3). In fact, with the new agent labels we can rewrite (2.2) into

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_l \end{bmatrix} = -\begin{bmatrix} L_f & l_{fl} \\ l_{fl}^T & L_l \end{bmatrix} \begin{bmatrix} x_f \\ x_l \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \tag{2.4}$$

where I is the $m\times m$ identity matrix. Taking the control input to be the state feedback $u = \begin{bmatrix} l_{fl}^T & L_l \end{bmatrix} \begin{bmatrix} x_f & x_l \end{bmatrix}^T + v$ with the new control input $v \in \mathbb{R}^m$, we have

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_l \end{bmatrix} = -\begin{bmatrix} L_f & l_{fl} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_f \\ x_l \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v.$$

This is equivalent to

$$\dot{x}_f = -L_f x_f + l_{fl} x_l \text{ and } \dot{x}_l = v. \tag{2.5}$$

Since controllable subspaces are invariant under state feedback, it holds that

$$\mathcal{K}(L, M) = \mathcal{K}(L_f, l_{fl}) \times \mathbb{R}^m.$$

Therefore, the network (2.2) is controllable *if and only if* the network (2.3) is controllable.

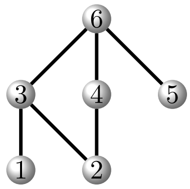

Figure 2.1: *Counterexample for [31, Thm. 3]*

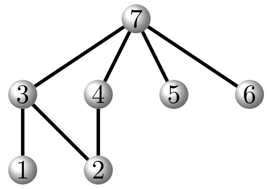

Figure 2.2: *Counterexample for [31, Prop. 2]*

2.3.2 Examples discussing sufficient conditions

Consider the graph depicted in Figure 2.1 where the sixth agent is the leader. By following the footsteps of [31], one obtains the dynamics of the network (given by [31, Eq. (3)]) as follows:

$$\dot{x} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 1 & 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u. \tag{2.6}$$

Straightforward calculations yield that the controllable subspace of the network (2.6) is of dimension 4, i.e. the network is uncontrollable. However, one can verify that the LEP is trivial. Hence, this example shows that the LEP being trivial is *not* a sufficient condition for controllability as [31, Thm. 3] claims.

The fallacy in the proof of [31, Thm. 3] is that Eq. (35) ([31, p. 118]) implies *only* the left hand side of Eq. (36) is included in the right hand side but *not* the reverse inclusion.

Now, consider the graph depicted in Figure 2.2 where the seventh agent is the leader. This leader-symmetric network leads to the dynamics (given by [31, Eq. (3)]):

$$\dot{x} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & 1 & -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u. \tag{2.7}$$

For this example, the controllable subspace is of dimension 4 and the LEP consists of the cells $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5, 6\}$, and $\{7\}$. As such, the controllable subspace is not equal to the span of the characteristic vectors corresponding to the cells $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5, 6\}$ as [31, Prop. 2] claims.

The fallacy in the proof of [31, Prop. 2] is similar to that of [31, Thm. 3]: The equation (16) ([31, p. 118]) implies *only* the controllable subspace is included in the span of the mentioned vectors but *not* the reverse inclusion. It should also be noted that [31, Prop. 2] deals with the leader-symmetric networks but this property was not used in the proof.

2.3.3 A class of uncontrollable networks: multi-chain topologies

In this subsection, we systematically construct a class of networks that act as further counterexamples for the result [31, Thm. 3], which implies that counterexamples for [31, Thm. 3] are *not* rare.

A simple undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, 2, \ldots, n\}$ is called a *path* graph if the edge set is $\mathcal{E} = \{(i, i+1) \text{ for } i = 1, 2, \ldots, n-1\}$, see e.g. [32, Ch. 2]. In the following, the vertex n is assigned to be the leader whenever we consider a network (2.3) with a topology described by a path graph $\mathbb{G}$.

Lemma 2.3.1 *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two simple undirected path graphs with n_1 and n_2 vertices, respectively. Let λ be an eigenvalue of $L_f(\mathbb{G}_1)$. Then λ is an eigenvalue of $L_f(\mathbb{G}_2)$ where $n_2 = k(2n_1 - 1) + n_1$ for any nonnegative integer k.*

Proof. Let $v = \begin{bmatrix} v_1 & v_2 & \ldots & v_{n_1} \end{bmatrix}^T$ be the eigenvector associated with the eigenvalue λ of $L_f(\mathbb{G}_1)$. Construct a vector $\bar{v} \in \mathbb{R}^{k(2n_1-1)+n_1-1}$ as follows:

$$\bar{v}_j = \begin{cases} v_1 & \text{if } j \bmod 2(2n_1-1) = 1 \text{ or } 2n_1 - 2 \\ v_2 & \text{if } j \bmod 2(2n_1-1) = 2 \text{ or } 2n_1 - 3 \\ \vdots & \vdots \\ v_{n_1} & \text{if } j \bmod 2(2n_1-1) = n_1 - 1 \text{ or } n_1 \\ 0 & \text{if } j \bmod (2n_1-1) = 0 \\ -v_1 & \text{if } j \bmod 2(2n_1-1) = 2n_1 \text{ or } 4n_1 - 3 \\ -v_2 & \text{if } j \bmod 2(2n_1-1) = 2n_1 + 1 \text{ or } 4n_1 - 4 \\ \vdots & \vdots \\ -v_{n_1} & \text{if } j \bmod 2(2n_1-1) = 3n_1 - 2 \text{ or } 3n_1 - 1 \end{cases}$$

Then one can check that $L_f(\mathbb{G}_2)\bar{v} = \lambda\bar{v}$, i.e., λ is an eigenvalue of $L_f(\mathbb{G}_2)$. ∎

For a positive integer l, a simple undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ is said to be an *l-path* graph if there exist integers $1 \leqslant r_1 < r_2 < \ldots < r_{l-1} < r_l = n - 1$ such that E is the union of the edge sets $\{(r_i, n) \text{ for all } i = 1, 2, \ldots, l\}$ and $\{(j, j+1), 1 < j < n-1 \text{ and } j \neq r_1, \ldots, r_{l-1}\}$. Intuitively, an l-path graph $\mathbb{G}$ consists of l paths that share a common end vertex n. Denote by $\mathbb{G}_i$ the path containing vertex r_i for

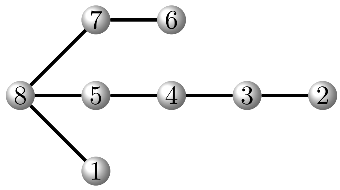

Figure 2.3: *A* 3*-path graph with* 8 *vertices*

$i = 1, 2, \ldots, l$. By definition, $\mathbb{G}_i$ has $r_i - r_{i-1} + 1$ vertices where $i \in \{1, 2, \ldots, l\}$ and r_0 is defined to be 0. Moreover,

$$L_{\text{f}}(\mathbb{G}) = \text{blockdiag}(L_{\text{f}}(\mathbb{G}_1), L_{\text{f}}(\mathbb{G}_2), \ldots, L_{\text{f}}(\mathbb{G}_l)). \tag{2.8}$$

As an example, a 3-path graph is depicted in Figure 2.3 that has 8 vertices with $r_1 = 1$, $r_2 = 5$ and $r_3 = 7$.

Lemma 2.3.2 *If $\mathbb{G}$ is an l-path graph where l_0 of the l paths have $3k_i + 2$ vertices for some nonnegative integer k_i and $i \in \{1, 2, \ldots, m\}$, then -1 is one eigenvalue of the matrix $L_f(\mathbb{G})$ with its (geometric) multiplicity being at least l_0.*

Proof. Observe that -1 is the unique eigenvalue of $L_{\text{f}}(P)$ if P is a path graph with 2 vertices. By setting $n_1 = 2$, it follows from Lemma 2.3.1 that -1 is an eigenvalue of $L_{\text{f}}(\mathbb{G}_i)$ for each $\mathbb{G}_i$ that has $3k_i + 2$ vertices for some nonnegative integer k_i. Then the conclusion follows from the block diagonal structure (2.8) of $L_{\text{f}}(\mathbb{G})$. ■

Proposition 2.3.3 *The network* (2.3) *with an l-path topology defined as in Lemma 2.3.2 is not controllable with $V_{\text{L}} = \{n\}$ if $l_0 > 1$.*

The proof of this proposition makes use of the following result.

Lemma 2.3.4 *[32, Prop. 10.3] The network* (2.3) *is uncontrollable if $L_f(\mathbb{G})$ has an eigenvalue whose geometric multiplicity is greater than one.*

Proof of Proposition 2.3.3. The conclusion follows directly from Lemma 2.3.4 since $L_{\text{f}}(\mathbb{G})$ has an eigenvalue -1 whose geometric multiplicity is at least l as proven in Lemma 2.3.2. ■

If $\mathbb{G}$ is an l-path graph as in Lemma 2.3.2 with k_i's different from each other for $i = 1, 2, \ldots, l$, then an algorithm developed in [15] tells that all the cells of LEP ([31, Def. 5]) in $\mathbb{G}$ are singleton. As such, each network of the class provided in

Proposition 2.3.3 acts as a counterexample of [31, Thm. 3]. For instance, the network (2.3) with its topology described by the graph in Figure 2.3 is such a counterexample where $l_0 = 2$.

By means of counterexamples in subsections 2.3.2 and 2.3.3, we showed that the results in [31, Prop. 2] and [31, Thm. 3] are erroneous. Also, we pointed out the fallacies in the proofs of these results. The counterexamples suggest that the controllability properties of single leader networks studied by [31] cannot be completely characterized by the LEP. As such, finding necessary and sufficient conditions for the controllability of such networks in terms of graph is still an open problem. This motivates our study in the next section.

2.4 Controllability of networks and graph partitions

In this section, we focus on networks where each agent has single-integrator dynamics in order to infer network controllability from its topology. As in (2.2), dynamics of such networks are as follows:

$$\dot{x} = -Lx + Mu.$$

Controllability of such networks is completely determined by the underlying topology given by the pair of matrices L and M. In what follows, we want to provide lower and upper bounds for the controllable subspace $\mathcal{K} = \operatorname{im} M + L \operatorname{im} M + \cdots + L^{n-1} \operatorname{im} M$ of the network (2.2). Towards this end, we quickly review graph partitions.

2.4.1 Graph partitions

Let $\mathbb{G}$ be an undirected graph with the vertex set $\mathcal{V}$. A subset $\mathcal{C}$ of $\mathcal{V}$ is called a *cell.* A collection of cells $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ is called a *partition* if the cells are mutually disjoint and $\bigcup_i \mathcal{C}_i = \mathcal{V}$. We use $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ to denote the partition. The *characteristic matrix* $P(\pi) \in \mathbb{R}^{n \times k}$ of the partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ is defined by

$$P_{ij}(\pi) \triangleq \begin{cases} 1 & \text{if } i \in \mathcal{C}_j \\ 0 & \text{otherwise.} \end{cases}$$

Next, we introduce particular partitions and employ them in order to obtain bounds for the controllable subspace of the network (2.2).

2.4.2 Lower bounds by distance partitions

The *distance* between two vertices $i, j \in \mathcal{V}$ is the length of the shortest path from i to j in $\mathbb{G}$ and will be denoted by $\operatorname{dist}(i,j)$. For convenience, we say $\operatorname{dist}(i,i)=0$ for any $i \in \mathcal{V}$. The *diameter* of $\mathbb{G}$ is defined by $\operatorname{diam}(\mathbb{G}) \triangleq \max_{i,j\in\mathcal{V}} \operatorname{dist}(i,j)$. Obviously, when $\mathbb{G}$ is connected [32] and $n>1$, it holds that $0<\operatorname{diam}(\mathbb{G})<n$. Let $\mathbb{G}$ be a connected graph and $v \in \mathcal{V}$. The *distance partition relative to* v consists of the cells $\mathcal{C}_i=\{w\in\mathcal{V} \mid \operatorname{dist}(w,v)=i\}$ for $0\leqslant i\leqslant \operatorname{diam}(\mathbb{G})$. We denote the distance partition relative to v by $\pi_{\mathrm{D}}(v)$. The following lemma is a direct consequence of the definition of $\pi_{\mathrm{D}}(v)$.

Lemma 2.4.1 *Let $\mathcal{C}_i$ and $\mathcal{C}_j$ be two cells of a distance partition relative to a certain vertex. For any $z\in\mathcal{C}_i$ and $w\in\mathcal{C}_j$, it holds that $|i-j|\leqslant \operatorname{dist}(z,w)$.*

The following theorem provides a lower bound for the dimension of the controllable subspace $\mathcal{K}$ in terms of the distance partition relative to the leaders.

Theorem 2.4.2 *If $\mathbb{G}$ is connected then $\dim\mathcal{K}\geqslant \max_{v_i\in\mathcal{V}_L}\operatorname{card}(\pi_D(v_i))$.*

Proof. We first prove that if $\mathcal{V}_{\mathrm{L}}=\{v\}$ then $\dim\mathcal{K}\geqslant\operatorname{card}(\pi_{\mathrm{D}}(v))$. Without loss of generality, we can take $v=1$, $\pi_{\mathrm{D}}(1)=\{\mathcal{C}_0,\mathcal{C}_1,\ldots,\mathcal{C}_r\}$ with $r\leqslant\operatorname{diam}\mathbb{G}$,

$$\mathcal{C}_0=\{1\}, \text{ and}$$
$$\mathcal{C}_q=\{i_q+1, i_q+2,\ldots,i_{q+1}\}$$

where $1\leqslant q\leqslant r$ and $1=i_1<i_2<\cdots<i_{r+1}=n$. In view of Lemma 2.4.1, we know that no vertex in $\mathcal{C}_i$ has a neighbor in $\mathcal{C}_j$ if $|i-j|>1$. This means that L is of the form

$$L=\begin{bmatrix} \deg(1) & \mathbf{1}^T & 0 & 0 & \cdots & 0 & 0 & 0\\ \mathbf{1} & L_{11} & L_{12} & 0 & \cdots & 0 & 0 & 0\\ 0 & L_{21} & L_{22} & L_{23} & \cdots & 0 & 0 & 0\\ 0 & 0 & L_{32} & L_{33} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & L_{r-2,r-2} & L_{r-2,r-1} & 0\\ 0 & 0 & 0 & 0 & \cdots & L_{r-1,r-2} & L_{r-1,r-1} & L_{r-1,r}\\ 0 & 0 & 0 & 0 & \cdots & 0 & L_{r,r-1} & L_{r,r} \end{bmatrix}, \tag{2.9}$$

where $\mathbf{1}$ is the all-one column vector of dimension $\operatorname{card}(\mathcal{C}_1)$ and L_{kl} are $\operatorname{card}(\mathcal{C}_k)\times\operatorname{card}(\mathcal{C}_l)$ matrices for all $1\leqslant k,l\leqslant r$. Since $v=1$, $M=e_1=\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}^T$. Let $E=\begin{bmatrix}e_1 & Le_1 & \cdots & L^r e_1\end{bmatrix}$. Then, we get

$$E=\begin{bmatrix} 1 & \deg(1) & * & * & \cdots & * & *\\ 0 & \mathbf{1} & * & * & \cdots & * & *\\ 0 & 0 & L_{21}\mathbf{1} & * & \cdots & * & *\\ 0 & 0 & 0 & L_{32}L_{21}\mathbf{1} & \cdots & * & *\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & L_{r-1,r-2}\ldots L_{21}\mathbf{1} & *\\ 0 & 0 & 0 & 0 & \cdots & 0 & L_{r,r-1}\ldots L_{21}\mathbf{1} \end{bmatrix},$$

where '*' denotes the corresponding elements of less interest. Since the graph $\mathbb{G}$ is connected, each diagonal block must be nonzero. Then $\operatorname{rank} E = \operatorname{card}(\pi_{\mathrm{D}}(1)) = r + 1$. Therefore, by further using the fact that $r \leqslant \operatorname{diam} \mathbb{G} \leqslant n - 1$, we have

$$\operatorname{card}(\pi_{\mathrm{D}}(v)) = \operatorname{rank} E \leqslant \operatorname{rank} \begin{bmatrix} e_1 & Le_1 & \cdots & L^{n-1}e_1 \end{bmatrix} = \dim \mathcal{K}.$$

Now we consider the case when $\mathcal{V}_L = \{v_1, v_2, \ldots, v_m\}$. Clearly, we have $\dim \mathcal{K} \geqslant \operatorname{card}(\pi_{\mathrm{D}}(v_i))$ for any $v_i \in \mathcal{V}_L$. Therefore, $\dim \mathcal{K} \geqslant \max_{v_i \in \mathcal{V}_{\mathrm{L}}} \operatorname{card}(\pi_{\mathrm{D}}(v_i))$. ■

Next, we introduce almost equitable partitions in order to provide an upper bound for the controllable subspace.

2.4.3 Upper bounds by almost equitable partitions

For a graph $\mathbb{G}$, a partition $\pi = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ is said to be an *almost equitable partition* if for any distinct cells $\mathcal{C}_i$ and $\mathcal{C}_j$, every vertex in $\mathcal{C}_i$ has the same number of neighbors in $\mathcal{C}_j$ [16]. We denote the set of all the almost equitable partitions of $\mathbb{G}$ by Π_{AEP}. Almost equitable partitions have the following key property that is related to the Laplacian matrices of the corresponding graphs. In the sequel, we say that a subspace $\mathcal{X} \subseteq \mathbb{R}^n$ is T*-invariant* if $T\mathcal{X} \subseteq \mathcal{X}$ where $T : \mathbb{R}^n \to \mathbb{R}^n$.

Lemma 2.4.3 *[16, Prop. 1] A partition π is almost equitable if and only if $\operatorname{im} P(\pi)$ is L-invariant.*

To come up with an upper bound for the controllable subspace, we need to compare different partitions. We say that a partition π_1 is *finer* than another partition π_2 if each cell of π_1 is a subset of some cell of π_2 and we write $\pi_1 \leqslant \pi_2$. It is a direct consequence of the definition that

$$\pi_1 \leqslant \pi_2 \quad \Longleftrightarrow \quad \operatorname{im} P(\pi_2) \subseteq \operatorname{im} P(\pi_1). \tag{2.10}$$

Let $\pi_{\mathrm{L}} = \{\{v_1\}, \{v_2\}, \ldots, \{v_m\}, \mathcal{V} \setminus \mathcal{V}_{\mathrm{L}}\}$. Note that $\operatorname{im} M \subseteq \operatorname{im} P(\pi_{\mathrm{L}})$. Define

$$\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}}) = \{\pi \mid \pi \in \Pi_{\mathrm{AEP}} \text{ and } \pi \leqslant \pi_{\mathrm{L}}\}. \tag{2.11}$$

The following theorem shows that each partition belonging to $\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ provides an upper bound for the controllable subspace $\mathcal{K}$.

Theorem 2.4.4 *For any $\pi \in \Pi_{AEP}(\pi_L)$, $\mathcal{K} \subseteq \operatorname{im} P(\pi)$.*

Proof. It follows from Lemma 2.4.3 that $\operatorname{im} P(\pi)$ is L-invariant for each $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$. As noted before, $\operatorname{im} M \subseteq \operatorname{im} P(\pi_{\mathrm{L}})$. In view of (2.10), this means that $\operatorname{im} M \subseteq \operatorname{im} P(\pi)$

for each π with $\pi \leqslant \pi_{\mathrm{L}}$. In particular, we have $\operatorname{im} M \subseteq \operatorname{im} P(\pi)$ for each $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$. Therefore, the subspace $\operatorname{im} P(\pi)$ is L-invariant and contains $\operatorname{im} M$. Since the controllable subspace $\mathcal{K}$ is the smallest subspace with these properties, we get $\mathcal{K} \subseteq \operatorname{im} P(\pi)$ for each $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$. ■

Remark 2.4.5 Theorem 2.4.4 applies when there are multiple leaders. As such, it extends the similar result of [31, Prop. 2] (see also [13]) which deals with the single leader case.

A natural question to ask is how to sharpen the upper bounds provided by Theorem 2.4.4. Obviously, the tightest bound which can be obtained by this theorem is given by

$$\mathcal{K} \subseteq \bigcap_{\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})} \operatorname{im} P(\pi).$$

However, this bound is not very practical as it requires the computation of all almost equitable partitions which are finer than the partition π_{L}. The relation (2.10) suggests that one can provide an upper bound in terms of a partition which is maximal in a certain sense. More precisely, if one can show that there exists a partition $\pi^* \in \Pi_{\mathrm{AEP}}$ such that $\pi \leqslant \pi^*$ for each $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ then one can conclude that

$$\mathcal{K} \subseteq \operatorname{im} P(\pi^*).$$

In [31] (see also [13]), such a bound is provided for the single leader case without formally proving the existence of such a partition. In what follows, we investigate the structure of the set Π_{AEP} in detail and show that such a maximal partition exists and is unique. Furthermore, we will present an algorithm in order to compute this maximal partition. To do so, we need to introduce some nomenclature.

Let Π denote the set of all the partitions of $\mathbb{G}$. With the partial order "$\leqslant$", the set Π becomes a complete lattice (see e.g. [11]) which means that every subset of Π has both its greatest lower bound and least upper bound within Π. We use $\vee\Pi'$ to denote the least upper bound of a subset Π'. By definition, the least upper bound $\vee\Pi'$ has the following property:

$$\pi \leqslant \tilde{\pi} \quad \text{for all } \pi \in \Pi' \qquad \Longrightarrow \qquad \vee\Pi' \leqslant \tilde{\pi}. \tag{2.12}$$

The complete lattice structure of Π readily implies that the set $\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ admits a unique least upper bound $\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}}) = \vee\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ such that $\pi \leqslant \pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ for each $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$. However, the least upper bound of a subset of Π does not need to belong to the subset in general. As such, one needs to further show that $\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ belongs to $\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ in order to conclude that $\mathcal{K} \subseteq \operatorname{im} P(\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}}))$. To do so, we first state the following auxiliary lemma.

Lemma 2.4.6 *[63, Lem. 1] For any subset Π' of Π, it holds that*

$$\bigcap_{\pi\in\Pi'} \operatorname{im} P(\pi) = \operatorname{im} P(\bigvee \Pi').$$

Based on this lemma, we are in a position to prove that the maximal partition $\pi^*_{\text{AEP}}(\pi_{\text{L}})$ belongs to the set $\Pi_{\text{AEP}}(\pi_{\text{L}})$.

Lemma 2.4.7 *It holds that $\pi^*_{AEP}(\pi_L) \in \Pi_{AEP}(\pi_L)$.*

Proof. It follows from (2.11) and (2.12) that $\pi^*_{\text{AEP}}(\pi_{\text{L}}) \leqslant \pi_{\text{L}}$. Therefore, it remains to show that $\pi^*_{\text{AEP}}(\pi_{\text{L}})$ is an almost equitable partition. To see this, note that

$$\operatorname{im} P(\pi^*_{\text{AEP}}(\pi_{\text{L}})) = \operatorname{im} P(\bigvee \Pi_{\text{AEP}}(\pi_{\text{L}})) = \bigcap_{\pi\in\Pi_{\text{AEP}}(\pi_{\text{L}})} \operatorname{im} P(\pi) \tag{2.13}$$

due to Lemma 2.4.6. In view of Lemma 2.4.3, $\operatorname{im} P(\pi)$ is L-invariant for each $\pi \in \Pi_{\text{AEP}}(\pi_{\text{L}})$. Since intersection of L-invariant subspaces must be L-invariant too, it follows from (2.13) that $\operatorname{im} P(\pi^*_{\text{AEP}}(\pi_{\text{L}}))$ is L-invariant. As such, one can conclude from Lemma 2.4.3 that $\pi^*_{\text{AEP}}(\pi_{\text{L}})$ is an almost equitable partition. ■

Combining Theorem 2.4.4 with Lemma 2.4.7, we can state the following tightened bound for the controllable subspace.

Theorem 2.4.8 *It holds that $\mathcal{K} \subseteq \operatorname{im} P(\pi^*_{AEP}(\pi_L))$.*

Remark 2.4.9 The bounds presented in Theorem 2.4.2 and Theorem 2.4.8 are tight for general graphs in the sense that one can construct graphs such that those bounds hold with equality. Consider the network (2.2) associated with the graph depicted on the left of Figure 2.4. If agents 1 and 4 are chosen as leaders, then the lower bound holds with equality which is strictly less than the upper bound. If agents 1 and 3 are chosen as leaders, then the upper bound holds with equality which is strictly greater than the lower bound. For the network associated with the graph shown on the right of Figure 2.4, if we choose agent 1 to be the single leader, neither of the two bounds is achieved. Moreover, one can check that there is no partition for which the image of its characteristic matrix is equal to the controllable subspace.

The bounds in Theorems 2.4.2 and 2.4.8 coincide for some specific graphs, for instance distance regular graphs with a single leader as shown in [63].

The lower bound in Theorem 2.4.2 is easy to check since distance partitions can be obtained rather straightforwardly. However, the computation for the upper bound presented in Theorem 2.4.8 is not so straightforward since there are no algorithms to

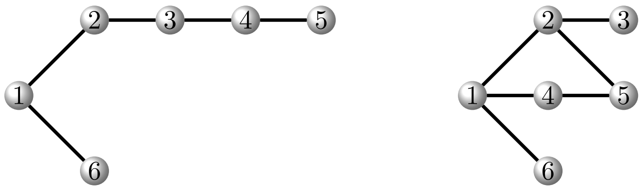

Figure 2.4: *Examples illustrating the tightness of the lower/upper bounds*

obtain almost equitable partitions with the constraint that multiple cells (corresponding to the leaders in our setting) have been strictly specified. In the next section, we develop an algorithm through which the least upper bound $\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ of the set $\Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ can be computed starting from a given partition π_{L}.

2.4.4 Algorithm to compute $\pi^*_{\mathbf{AEP}}(\pi_{\mathbf{L}})$

To present the algorithm, we need to define a few concepts first. Let $\mathbb{R}^{n\times\bullet}$ denote all matrices with n rows. Let $\psi : \mathbb{R}^{n\times\bullet} \to \Pi$ be the mapping such that for any matrix $X \in \mathbb{R}^{n\times\bullet}$ it holds that i and j are in the same cell of $\psi(X)$ if and only if the ith and jth rows of the matrix X are the same. Note that

$$\pi = \psi(P(\pi)) \tag{2.14}$$

for any partition $\pi \in \Pi$.

Now we present an algorithm that computes $\pi^*_{\mathrm{AEP}}(\pi_{\mathrm{L}})$ starting from partition π_{L}.

Theorem 2.4.10 *Define the sequence $\{pi_k\}$ of partitions by*

$$\begin{aligned} \pi_0 &= \pi_L \\ \pi_{k+1} &= \psi(\begin{bmatrix} P(\pi_k) & LP(\pi_k) \end{bmatrix}) \end{aligned} \tag{2.15}$$

where $k \geqslant 0$. Then, there exists an integer q with $0 \leqslant q \leqslant n-m$ such that $\pi_q = \pi^*_{AEP}(\pi_L) = \pi_{q+\ell}$ for all $\ell \geqslant 0$.

To prove this theorem, we will use the following auxiliary lemma.

Lemma 2.4.11 *Let $X, Y \in \mathbb{R}^{n\times\bullet}$. The following statements hold.*

1. $\operatorname{im} X \subseteq \operatorname{im} P(\psi(X))$.
2. $\operatorname{im} X \subseteq \operatorname{im} Y$ *implies that* $\operatorname{im} P(\psi(X)) \subseteq \operatorname{im} P(\psi(Y))$.

Proof. 1): It follows from the definition of ψ that for each matrix $X \in \mathbb{R}^{n\times\bullet}$ there exists a matrix Z_X such that $X = P(\psi(X))Z_X$. Consequently, $\operatorname{im} X \subseteq \operatorname{im} P(\psi(X))$.

2): In view of (2.10), it suffices to prove that $\psi(Y) \leqslant \psi(X)$. To do so, let i and j be such that the ith and jth rows of the matrix Y are the same. Since $\operatorname{im} X \subseteq \operatorname{im} Y$, there exists a matrix Z such that $X = YZ$. Then, the ith and jth rows of the matrix X must the same. Therefore, it follows from the definition of ψ that any cell of $\psi(Y)$ is a subset of a cell of $\psi(X)$. In other words, $\psi(Y) \leqslant \psi(X)$. ∎

Now we are ready to prove Theorem 2.4.10.

Proof of Theorem 2.4.10. Note that

$$\operatorname{im} P(\pi_k) \subseteq \operatorname{im} \begin{bmatrix} P(\pi_k) & LP(\pi_k) \end{bmatrix}.$$

Then, it follows from Lemma 2.4.11.2, (2.10) and (2.14) that

$$\pi_{k+1} = \psi(\begin{bmatrix} P(\pi_k) & LP(\pi_k) \end{bmatrix}) \leqslant \psi(P(\pi_k)) = \pi_k.$$

Therefore, we obtain

$$\pi_{k+1} \leqslant \pi_k \tag{2.16}$$

for all $k \geqslant 0$. Now, we claim that the implication

$$\pi_{r+1} = \pi_r \text{ for some } r \implies \pi_r = \pi_{r+\ell} \text{ for all } \ell \geqslant 0 \tag{2.17}$$

holds. To show this, note that $\pi_r = \psi(\begin{bmatrix} P(\pi_r) & LP(\pi_r) \end{bmatrix})$ if $\pi_{r+1} = \pi_r$. Then, it follows from Lemma 2.4.11.1 and (2.14) that $\operatorname{im} \begin{bmatrix} P(\pi_r) & LP(\pi_r) \end{bmatrix} \subseteq \operatorname{im} P(\pi_r)$. This means that

$$\operatorname{im} LP(\pi_r) \subseteq \operatorname{im} P(\pi_r). \tag{2.18}$$

Since $\pi_{r+1} = \pi_r$, (2.18) implies that $\pi_{r+\ell} = \pi_r$ for all $\ell \geqslant 0$. Since $\operatorname{card}(\pi_{\mathrm{L}}) = m + 1$ when $m < n$ and $\operatorname{card}(\pi_{\mathrm{L}}) = m$ when $m = n$, we get $\operatorname{card}(\pi_0) \geqslant m$. Then, (2.16) and the implication (2.17) imply that there exists an integer q with $0 \leqslant q \leqslant n - m + 1$ such that $\pi_q = \pi_{q+\ell}$ for all $\ell \geqslant 0$. What remains to prove is that

$$\pi_q = \pi^*_{\mathrm{AEP}}(\pi_0). \tag{2.19}$$

From (2.18), we know that $P(\pi_q)$ is L-invariant. Then, π_q is an almost equitable partition due to Lemma 2.4.3. We also know from (2.16) that $\pi_q \leqslant \pi_0$. Therefore, $\pi_q \in \Pi_{\mathrm{AEP}}(\pi_0)$. This implies that

$$\pi_q \leqslant \pi^*_{\mathrm{AEP}}(\pi_0). \tag{2.20}$$

Now, we claim that

$$\pi^*_{\text{AEP}}(\pi_0) \leqslant \pi_k \tag{2.21}$$

for each $k \geqslant 0$. We prove this claim by induction on k. When $k = 0$, (2.21) follows from the definition of $\pi^*_{\text{AEP}}(\pi_0)$ that $\pi^*_{\text{AEP}}(\pi_0) \leqslant \pi_0$. Now, assume that $\pi^*_{\text{AEP}}(\pi_0) \leqslant \pi_k$ holds for some $k \geqslant 0$. It follows from (2.10) that $\operatorname{im} P(\pi_k) \subseteq \operatorname{im} P(\pi^*_{\text{AEP}}(\pi_0))$ and from Lemma 2.4.3 that $L \operatorname{im} P(\pi_k) \subseteq \operatorname{im} P(\pi^*_{\text{AEP}}(\pi_0))$, both of which imply that $\operatorname{im}\begin{bmatrix} P(\pi_k) & LP(\pi_k)\end{bmatrix} \subseteq \operatorname{im} P(\pi^*_{\text{AEP}}(\pi_0))$. Then, we obtain from Lemma 2.4.11.2 and (2.14) that $\operatorname{im} P(\pi_{k+1}) \subseteq \operatorname{im} P(\pi^*_{\text{AEP}}(\pi_0))$. Hence, (2.10) yields that $\pi^*_{\text{AEP}}(\pi_0) \leqslant \pi_{k+1}$. Consequently, (2.21) is proven. In particular, we can conclude that

$$\pi^*_{\text{AEP}}(\pi_0) \leqslant \pi_q.$$

Together with (2.20), this implies that (2.19) holds. ■

To illustrate the algorithm by means of examples, we consider the diffusively coupled network (2.2) corresponding to the graph depicted in Figure 2.5. Note that the Laplacian matrix is given by

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 6 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 4 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 4 \end{bmatrix}$$

for this graph. We employ the algorithm for three different leader sets. Figures 2.6, 2.7, and 2.8 depict, respectively, the partitions obtained by the recursion (2.15) for the leader sets $\mathcal{V}_{\text{L}} = \{1\}$, $\mathcal{V}_{\text{L}} = \{4\}$, and $\mathcal{V}_{\text{L}} = \{1, 4\}$. In all these cases, the last partition correspond to the maximal almost equitable partition $\pi^*_{\text{AEP}}(\pi_{\text{L}})$.

2.5 Controllability of networks: distance regular topologies

As discussed in Remark 2.4.9, the lower and upper bounds do not coincide in general. In this section, we focus on a particular class of graphs, namely distance regular graphs. For such graphs, we will first show that the lower and upper bounds coincide

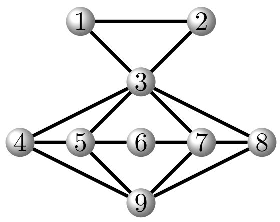

Figure 2.5: *Example for the algorithm*

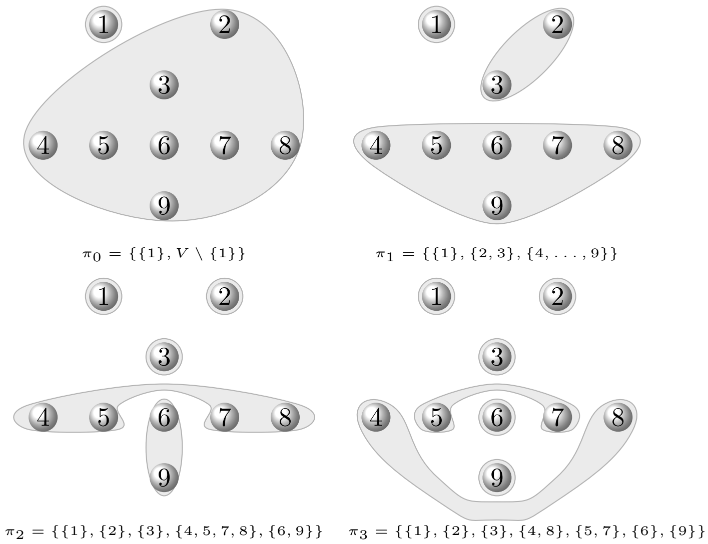

Figure 2.6: *Execution of the algorithm for* $\mathcal{V}_L = \{1\}$

if there is a single leader. For multi-leader scenarios, we will provide a systematic way of choosing leaders in order to render the network to be controllable by further exploiting the distance regularity property.

We begin with a brief review of the properties of distance regular graphs. For more details, readers can refer to [8].

2.5.1 Distance regular graphs and properties

A graph $\mathbb{G}$ is said to be *regular* if $\deg(i) = \deg(j)$ for all $i,\ j \in \mathcal{V}$, where $\deg(i)$ denotes the number of neighbors of vertex i in $\mathbb{G}$. It is called *distance-regular* if it is regular

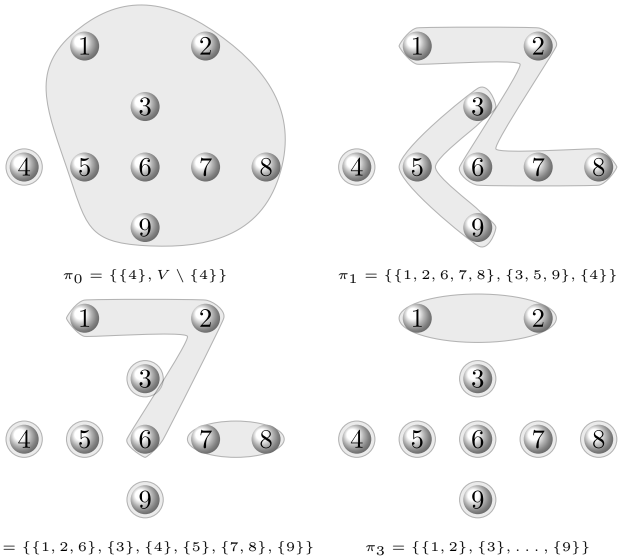

Figure 2.7: *Execution of the algorithm for* $\mathcal{V}_L = \{4\}$

and for any pair of vertices $u, v \in \mathcal{V}$ with $\mathrm{dist}(u, v) = i$, $0 < i < \mathrm{diam}(\mathbb{G})$, there exist numbers c_i and b_i such that there are c_i neighbors of v that are of distance $i-1$ from u and b_i neighbors of v that are of distance $i+1$ from u [8].

Consider a distance regular graph $\mathbb{G}$. Let $d = \mathrm{diam}(\mathbb{G})$. The sequence

$$\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$$

is called the *intersection array* of $\mathbb{G}$. For a pair of vertices $u, v \in \mathcal{V}$ with $\mathrm{dist}(u, v) = h$, we define the numbers $p_{ij}^h \triangleq \mathrm{card}(\{w \in \mathcal{V} | \, \mathrm{dist}(u, w) = i \text{ and } \mathrm{dist}(v, w) = j\})$ for all $0 \leqslant i, j, h \leqslant d$. Then we have

Lemma 2.5.1 *[8] For a distance regular graph, it holds that*

$$b_{i-1} > 0, c_i > 0 \quad \textit{for all } 1 \leqslant i \leqslant d$$
$$p_{ih}^{i-h} = \frac{b_{i-1} \cdots b_{i-h}}{c_1 \cdots c_h} \quad \textit{for all } 0 \leqslant h \leqslant i \leqslant d.$$

Define the i distance regular graph of $\mathbb{G}$ for $i = 0, 1, \dots, d$, denoted by $\mathbb{G}_i = (\mathcal{V}, \mathcal{E}_i)$, such that for $u, v \in \mathcal{V}$, $(u, v) \in \mathcal{E}_i$ if and only if $\mathrm{dist}(u, v) = i$ in $\mathbb{G}$. Denote the adjacency matrix of $\mathbb{G}_i$ by A_i. Note that $A_0 = I$ and $A_1 = A$ where A denotes the

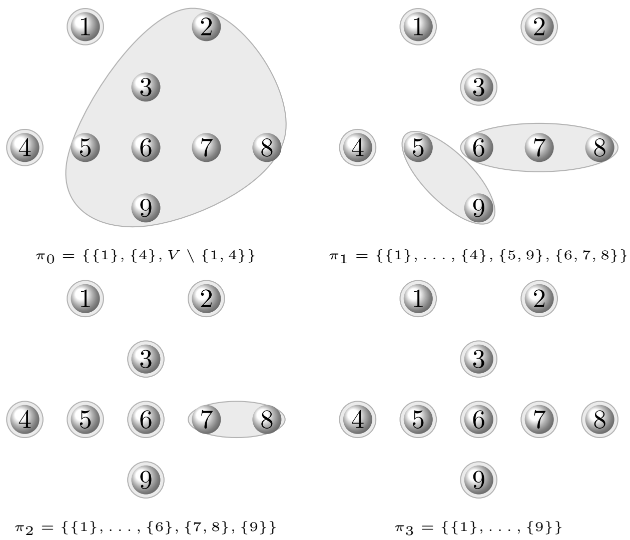

Figure 2.8: *Execution of the algorithm for* $\mathcal{V}_\mathrm{L} = \{1,4\}$

adjacency matrix of $\mathbb{G}$. These matrices satisfy

$$I + A_1 + \cdots + A_d = J \tag{2.22}$$

where J is the matrix of all 1's. Further, there exist ith degree matrix polynomials v_i such that

$$A_i = v_i(A). \tag{2.23}$$

Moreover, the matrices $\{I, A_1, \dots, A_d\}$ are linearly independent.

2.5.2 Single-leader cases

Consider the network (2.2) with a distance regular graph $\mathbb{G}$ and a single leader. When $\mathcal{V}_\mathrm{L} = \{v\}$ with $v \in \mathcal{V}$, we denote its controllable subspace by $\mathcal{K}(v)$. Then the dimension of $\mathcal{K}(v)$ can be completely characterized in terms of graph partitions as follows.

Proposition 2.5.2 *For any* $v \in \mathcal{V}$,

$$\dim(\mathcal{K}(v)) = \mathrm{card}(\pi_D(v)) = d + 1$$

where $d = \mathrm{diam}(\mathbb{G})$.

Proof. It follows from the definition of distance regularity that the distance partition $\pi_{\mathrm{D}}(v)$ is an almost equitable partition. Hence, we have $\pi_{\mathrm{D}}(v) \leqslant \pi^*_{\mathrm{AEP}}(v)$ and thus, $\mathrm{card}(\pi^*_{\mathrm{AEP}}(v)) \leqslant \mathrm{card}(\pi_{\mathrm{D}}(v))$. Therefore, it follows from Theorems 2.4.2 and 2.4.8 that $\dim \mathcal{K}(v) = \mathrm{card}(\pi_{\mathrm{D}}(v))$. Moreover, it follows from Lemma 2.5.1 that $p^0_{dd} > 0$. This implies that for each vertex v, there exists at least one other vertex u such that $\mathrm{dist}(v, u) = d$, which in turn implies the claim. ■

Proposition 2.5.2 implies that the network (2.2) with a distance regular graph cannot be controllable with a single leader unless $n = d+1$. This condition is satisfied if and only if the graph consists of two vertices and one edge. This observation motivates a further interest in studying networks with multiple leaders.

2.5.3 How many leaders are necessary?

Consider the network (2.2) with a distance regular graph $\mathbb{G}$ and multiple leaders. Let

$$N = \begin{bmatrix} A_d M & A_{d-1}M & \cdots & A_1 M & A_0 M \end{bmatrix}. \tag{2.24}$$

where A_ℓ's ($0 \leqslant \ell \leqslant d$) are defined earlier. Then we have the following result.

Proposition 2.5.3 *It holds that*

$$\mathrm{im}\, N = \mathcal{K}. \tag{2.25}$$

Proof. For a subspace $\mathcal{W} \subseteq \mathbb{R}^n$, we denote its orthogonal complement by $\mathcal{W}^\perp$. Let $z \in \mathbb{R}^n$.

$$\begin{aligned} z^T \in \mathcal{K}^\perp &\iff z^T L^k M = 0 \text{ for all } k = 0, 1, \ldots, n-1 \\ &\iff z^T A^k M = 0 \text{ for all } k = 0, 1, \ldots, n-1 \\ &\overset{(2.22)}{\iff} z^T A_\ell M = 0 \text{ for all } \ell = 0, 1, \ldots, d \\ &\overset{(2.23)}{\iff} z \in \ker N^T. \end{aligned}$$

The second relation follows from the regularity of the graph, i.e. $L = aI - A$ where $a = \deg(i)$ for an $i \in \mathcal{V}$. ■

Theorem 2.5.4 *The network* (2.2) *with a distance regular graph $\mathbb{G}$ is controllable only if the number of inputs m satisfies*

$$dm \geqslant n - 1 \tag{2.26}$$

where $d = \mathrm{diam}(\mathbb{G})$.

Proof. Since the network is controllable, we have $\dim(\mathcal{K}) = n$. Then, we get

$$\begin{aligned} n &= \dim(\mathcal{K}) \\ &\overset{(2.25)}{=} \operatorname{rank} N \\ &\overset{(2.22)}{=} \operatorname{rank} \begin{bmatrix} JM & A_{d-1}M & \cdots & A_1M & M \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} \mathbf{1} & A_{d-1}M & \cdots & A_1M & M \end{bmatrix} \end{aligned}$$

where $\mathbf{1}$ is the column vector of 1's with the dimension of n. It follows from linear independence of $\{I, A_1, \dots, A_d\}$ and (2.22) that

$$\operatorname{rank} \begin{bmatrix} A_{d-1}M & \cdots & A_1M & M \end{bmatrix} \geqslant n - 1.$$

Note that the matrix on the left has n rows and dm columns. Hence, we get $dm \geqslant n - 1$. ∎

2.5.4 How many leaders are sufficient?

We begin with a procedure of choosing leaders and later we will show that this procedure guarantees controllability. Consider $w \in \mathcal{V}$. Denote the distance partition relative to w by $\pi_{\mathrm{D}}(w) = \{\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_d\}$. For each $1 \leqslant \ell \leqslant d$, choose $w_\ell \in \mathcal{C}_\ell$. Define $\mathcal{V}_{\mathrm{F}}^*$ to be the set of all such w_ℓ. Let $\mathcal{V}' = \mathcal{V} \setminus \mathcal{V}_{\mathrm{F}}^*$. Then $w \in \mathcal{V}'$ and $\operatorname{card}(\mathcal{V}') = n - d$.

The main result of the subsection is stated as follows.

Theorem 2.5.5 *When $\mathbb{G}$ is distance regular, the network* (2.2) *is controllable if $\mathcal{V}_{\mathrm{L}} = \mathcal{V}'$.*

Proof. Let $z \in \mathcal{K}^\perp$. Then we have $z \in \ker N^T$ due to Proposition 2.5.3. This means that

$$z^T A_\ell M = 0$$

for all $\ell = 0, 1, \dots, d$. In particular, we have

$$z^T M = 0.$$

If $v \in \mathcal{V}_{\mathrm{L}}$ then the vth standard basis vector must be a column of M. This leads to

$$z_v = 0 \tag{2.27}$$

where z_v is the vth element of z and $v \in \mathcal{V}_{\mathrm{L}}$. We claim that the other components of z are zero too. To see this, we consider the relation

$$z^T A_\ell M = 0$$

with $\ell \in \{1, 2, \ldots, d\}$. Let q be the wth column of $A_\ell M$. Note that the jth element of q, $1 \leqslant j \leqslant n$, is

$$q_j = \begin{cases} 1 & \text{if } j \in \mathcal{C}_\ell \\ 0 & \text{otherwise.} \end{cases}$$

Since all elements of $\mathcal{C}_\ell$ except w_ℓ belong to $\mathcal{V}_{\mathrm{L}}$, it follows from (2.27) that

$$0 = z^T q = z_{w_\ell}.$$

This means that $z = 0$ and hence $\mathcal{K} = \mathbb{R}^n$. ■

We have the following result as a direct consequence of Theorem 2.5.5.

Corollary 2.5.6 *Every distance regular graph can be rendered controllable with $n - d$ leaders.*

In the next section, we will discuss how to choose leaders for two well-known classes of distance regular graphs: cycles and complete graphs.

2.5.5 Leader selection: cycles and complete topologies

Cycle and complete graphs are two classes of distance regular graphs. A graph $\mathbb{G}$ is a *cycle* if $\deg(i) = 2$ for all $i \in \mathcal{V}$ and is *complete* if $\deg(i) = n - 1$ for all i. In the following, $\mathbb{C}_n$ and $\mathbb{K}_n$ are used to denote a cycle and a complete graph with n vertices respectively. Note that $n \geqslant 3$ for any $\mathbb{C}_n$.

Consider the network (2.2) with $\mathbb{G}$ being $\mathbb{C}_n$. Let $d = \mathrm{diam}(\mathbb{C}_n)$. Note that $n = 2d$ or $n = 2d + 1$. From Proposition 2.5.2, we know that such a network can never be controllable by a single leader. Moreover, from Theorem 2.5.5, we know that it is controllable by d leaders when n is even and $d + 1$ leaders when n is odd. In addition, we have the following result.

Theorem 2.5.7 *The network* (2.2) *with $\mathbb{C}_n$ and two leaders is controllable if the two vertices corresponding to the leaders are adjacent.*

Proof. The two adjacent leaders are denoted by v_1 and v_2 and their controllable subspaces $\mathcal{K}(v_1)$ and $\mathcal{K}(v_2)$ respectively. We use $\mathcal{K}(v_1, v_2)$ to denote the joint controllable subspace of v_1 and v_2.

From Proposition 2.5.2, $\dim \mathcal{K}(v_1) = \dim \mathcal{K}(v_2) = d + 1$. When $\mathrm{dist}(v_1, v_2) = 1$ in $\mathbb{C}_n$, we have

$$\dim(\mathcal{K}(v_1) \cap \mathcal{K}(v_2)) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

The conclusion follows from the fact that $\dim \mathcal{K}(v_1, v_2) = \dim \mathcal{K}(v_1) + \dim \mathcal{K}(v_2) - \dim(\mathcal{K}(v_1) \cap \mathcal{K}(v_2))$. ■

As Theorem 2.5.7 suggests, the controllability of the network with two leaders associated with $\mathbb{C}_n$ depends on the distance between the two leaders. We illustrate this point by an example of $\mathbb{C}_6$. We label the vertices of the graph clockwise by 1 to 6 and choose vertices 1 and 4 to be the two leaders. Then $\dim(\mathcal{K}(u_1) + \mathcal{K}(u_2)) = 4 < 6$, which implies that the network is uncontrollable when the distance between the two leaders is 3.

When $\mathbb{G}$ is $\mathbb{K}_n$, we have the following result.

Theorem 2.5.8 *The network* (2.2) *with $\mathbb{K}_n$ is controllable if and only if at least $n-1$ agents are leaders.*

Proof. Since $\mathrm{diam}(\mathbb{K}_n) = 1$, the necessity and the sufficiency directly follow from Theorems 2.5.4 and 2.5.5, respectively. ■

2.6 Concluding Remarks

We have studied controllability of networks of agents with general linear dynamics. After investigating the effect of network topologies on controllability, we focused on network with agents having single-integrator dynamics. For this case, we have presented a lower bound for controllable subspace in terms of the distance partitions and an upper bound in terms of the maximal almost equitable partitions. To compute the upper bound, we have provided an algorithm that finds the maximal almost equitable partition for given leaders. In particular, if the graphs are distance regular, this characterization is complete when there is a single leader, and a necessary condition and a sufficient condition have been provided when multiple leaders are present. For networks associated with cycles and complete graphs, we have shown how to choose leaders to guarantee their controllability.

As future research directions, we are interested in studying controllability of multi-agent networks when their associated graphs are directed or time-varying. Also, we are interested in systematic ways of choosing leaders for other classes of distance regular graphs. Moreover, we envision that the use of ideas and notions of geometric control theory in the context of multi-agent networks would lead to graph topological interpretation of many other fundamental control theoretic problems.

Chapter 3

Controllability of diffusively coupled networks: switching topologies

3.1 Introduction

In Chapter 2, we studied controllability of diffusively coupled networks when their underlying graphs are undirected and time-independent. In particular, we employed two classes of graph partitions together with notions of geometric control theory to provide lower and upper bounds for controllable subspaces. In this chapter, we investigate controllability of networks with switching topologies. As will be shown in this chapter, the ideas employed in Chapter 2 that combines graph partitions and geometric control theory can be extended directly to networks with switching topologies. As a result, this extension leads to graph theoretical conditions for controllability.

In the next section, we begin with reviewing results on controllability of switched linear systems. The controllability of diffusively coupled networks with switching topologies is defined in the same way as that of switched linear systems. In Section 3.3, we introduce diffusively coupled networks with switching topologies and general linear dynamical agents. Moreover, we show the role that the switching topology plays in controllability of the overall network. Then we focus on inferring network controllability from its topology in terms of graph partitions. In Section 3.4, the partitions used in Chapter 2 are extended to networks with switching topologies. The main results are presented in Section 3.5.

3.2 Review: controllability of switched linear systems

Let $(A_i, B_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ with $i \in \{1, 2, \dots, p\}$ be given matrices. Let S be the set of *switching signals* defined as the set of right-continuous, piecewise constant functions $\sigma : \mathbb{R}_+ \to \{1, 2, \dots, p\}$ which have finitely many discontinuities on every finite interval.

Consider the switched linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and $\sigma \in S$ is a switching signal. Let $x^{\xi,\sigma,u}$ denote the unique state trajectory of the system (3.1) for the initial state ξ (i.e. $x^{\xi,\sigma,u}(0) = \xi$), the switching signal σ, and the input u.

We say that a state $\eta \in \mathbb{R}^n$ is

- *reachable* if there exist $\sigma \in S$, locally-integrable u, and a positive number T such that $x^{0,\sigma,u}(T) = \eta$,
- *controllable* if there exist $\sigma \in S$, locally-integrable u, and a positive number T such that $x^{\eta,\sigma,u}(T) = 0$.

Let R and C denote the sets of all reachable and controllable states, respectively. We say that the system (3.1) is

- *reachable* if $R = \mathbb{R}^n$,
- *controllable* if $C = \mathbb{R}^n$,
- *completely controllable* if $R = C = \mathbb{R}^n$.

As it is shown in [50], the sets of reachable and controllable states coincide and are subspaces. To elaborate more, let $\langle A \mid B\rangle$ denote the smallest A-invariant subspace that contains $\operatorname{im} B$. Similarly, let $\langle\{A_1, A_2, \dots, A_p\} \mid \{B_1, B_2, \dots, B_p\}\rangle$ denote the smallest subspace that is invariant under A_i and contains $\operatorname{im} B_i$ for all $i = \{1, 2, \dots, p\}$. Then, it has been shown in [49] that

$$R = C = \langle\{A_1, A_2, \dots, A_p\} \mid \{B_1, B_2, \dots, B_p\}\rangle.$$

Moreover, the following result is well-known in switched linear systems, see e.g. [49].

Lemma 3.2.1 *Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$. The system* (3.1) *is controllable if and only if the following implication holds:*

$$A_i^T\mathcal{V} \subseteq \mathcal{V} \textit{ and } \mathcal{V} \subseteq \ker B_i^T \textit{ for all } i \in \{1, 2, \dots, p\} \Longrightarrow \mathcal{V} = \{0\}.$$

In the sequel, we are interested in a particular kind of switched linear systems for which each subsystem is a diffusively coupled multi-agent networks.

3.3 Diffusively coupled networks: switching topologies

Let $G_i = (V, E_i)$ where $i \in \{1, 2, \ldots, p\}$ be p simple undirected graphs with a common vertex set $V = \{1, 2, \ldots, n\}$. For each graph G_i, consider two subsets of V: $V_{\mathrm{L}}^i = \{v_1^i, v_2^i, \ldots, v_{m_i}^i\}$ and $V_{\mathrm{F}}^i = V \setminus V_{\mathrm{L}}^i$. With each vertex j of G_i, we associate the (general) linear dynamics as follows:

$$\dot{x}_j(t) = Ax_j(t) + Cz_j^i(t) + Bu_\ell(t) \quad \text{if } j = v_\ell^i \in V_{\mathrm{L}}^i \tag{3.2a}$$

$$\dot{x}_j(t) = Ax_j(t) + Cz_j^i(t) \quad \text{if } j \in V_{\mathrm{F}}^i \tag{3.2b}$$

where $x_j \in \mathbb{R}^N$ represents the state of the agent (vertex) i, $u_\ell \in \mathbb{R}^s$ indicates the input to the agent $j = v_\ell^i$, $z_j^i \in \mathbb{R}^q$ is the diffusive coupling term for the agent j in G_i and all the matrices are of compatible dimensions. Here, the diffusive coupling term z_j^i is given by

$$z_j^i(t) = -K \sum_{(j,k) \in E_i} (x_j(t) - x_k(t)). \tag{3.2c}$$

where $K \in \mathbb{R}^{q \times N}$ indicates the coupling strengths. As in Chapter 2, the vertices in the set V_{L}^i are called *leaders* in G_i and those in V_{F}^i are *followers*.

Define $x = \mathrm{col}\begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}$ and $u = \mathrm{col}\begin{pmatrix} u_1 & u_2 & \ldots & u_m \end{pmatrix}$ where $m = \max_{\forall i \in \{1,2,\ldots,q\}} m_i$. With this definition, (3.2) can be re-written as

$$\dot{x}(t) = -\hat{L}_i x(t) + \hat{M}_i u(t) \tag{3.3}$$

where $\hat{L}_i = I \otimes A - L_i \otimes CK$ with $I \in \mathbb{R}^{n \times n}$ being identity matrix, L_i being the Laplacian matrix of G_i and $\hat{M}_i = M_i \otimes B$ with $M_i \in \mathbb{R}^{n \times m}$ defined as follows:

$$[M_i]_{j\ell} = \begin{cases} 1 & \text{if } j = v_\ell^i \in V_{\mathrm{L}}^i \\ 0 & \text{otherwise.} \end{cases}$$

For $i \in \{1, 2, \ldots, p\}$, the leader/follower diffusively coupled network with a switching topology among G_i's is described by

$$\dot{x}(t) = -\hat{L}_{\sigma(t)} x(t) + \hat{M}_{\sigma(t)} u(t) \tag{3.4}$$

where $\sigma : [0, \infty) \to \{1, 2, \ldots, p\}$ is a right-continuous, piecewise constant function with finitely many of discontinuities on any finite time interval.

In case each agent of the network has single-integrator dynamics, the network dynamics can be obtained by setting $A = 0$, $B = C = K = 1$ and $N = 1$ in (3.4) as follows:

$$\dot{x}(t) = -L_{\sigma(t)} x(t) + M_{\sigma(t)} u(t). \tag{3.5}$$

The following result reveals the relationship between controllability of (3.5) and that of (3.4).

Theorem 3.3.1 *The network* (3.4) *is controllable only if both the network* (3.5) *and the pair* $(A, \begin{bmatrix} C & B \end{bmatrix})$ *are controllable.*

Proof. Let $\mathcal{V} \subseteq \mathbb{R}^n$ be a subspace such that

$$L_i \mathcal{V} \subseteq \mathcal{V} \text{ and } \mathcal{V} \subseteq \ker M_i^T \tag{3.6}$$

for any $i \in \{1, 2, \ldots, p\}$. Let $\{w_1, w_2, \ldots, w_r\}$ and $\{y_1, y_2, \ldots, y_N\}$ be bases of $\mathcal{V}$ and $\mathbb{R}^N$, respectively. Consider the subspace $\mathcal{W} = \operatorname{span}\{w_k \otimes y_j \mid 1 \leqslant k \leqslant r \text{ and } 1 \leqslant j \leqslant N\}$. Then it follows from (3.6) that $\hat{L}_i^T \mathcal{W} \subseteq \mathcal{W}$ and $\mathcal{W} \subseteq \ker \hat{M}_i^T$. Since the network (3.4) is controllable, it follows from Lemma 3.2.1 that $\mathcal{W} = \{0\}$. This means that $\mathcal{V} = \{0\}$ and hence, it concludes from Lemma 3.2.1 that the network (3.5) is controllable.

Let $y \in \mathbb{R}^N$ be such that $A^T y = \mu w$ for some $\mu \in \mathbb{C}$ and $y^H \begin{bmatrix} C & B \end{bmatrix} = 0$. Consider $z = w \otimes y$ where $w \in \mathbb{R}^n$ and $w \neq 0$. Then it follows that $\hat{L}_i^T z = \mu z$ and $z^T \hat{M}_i = 0$. Since the network (3.4) is controllable, we get $z = 0$ by Lemma 3.2.1. As $w \neq 0$, this results in $y = 0$. Hence, the pair $(A, \begin{bmatrix} C & B \end{bmatrix})$ is controllable. ∎

Next, we focus on inferring controllability of the network (3.5) from its switching topology. As in Chapter 2, we employ graph partitions to provide bounds for the controllable subspace of (3.5).

3.4 Graph partitions: extension for switching topologies

Graph partitions have been introduced in Chapter 2. Here, we review the terminologies briefly and extend almost equitable partitions to be applicable to switching cases.

Any nonempty subset of the vertex set V is called a *cell*. A collection of mutually disjoint cells $\pi = \{C_1, \ldots, C_r\}$ is said to be a *partition* of V if $\bigcup_{j=1}^r C_j = V$. Let Π denote the set of all the partitions of V. A partition π_1 is said to be *finer* than a partition π_2, denoted by $\pi_1 \leqslant \pi_2$, if each cell of π_1 is a subset of some cell of π_2.

Let $\Pi_{\text{AEP}}(G)$ denote the set of all almost equitable partitions of a given graph $G = (V, E)$. For the collection of graphs $\{G_1, \ldots, G_p\}$, define

$$\Pi_{\text{AEP}}(\pi_1, \ldots, \pi_p) = \{\pi \mid \pi \in \Pi_{\text{AEP}}(G_i) \text{ and } \pi \leqslant \pi_i \text{ for all } i\} \tag{3.7}$$

where π_i is a given partition of G_i. Denote by $\pi^*_{\text{AEP}}(\pi_1, \ldots, \pi_p)$ the least upper bound of the set (3.7). Then it can be proven similarly to Lemma 2.4.7 that

$$\pi^*_{\text{AEP}}(\pi_1, \ldots, \pi_p) \in \Pi_{\text{AEP}}(\pi_1, \ldots, \pi_p). \tag{3.8}$$

3.5 Main results

The main results of this chapter are lower and upper bounds for the controllable subspace of network (3.5). Let $\mathcal{K}$ denote the controllable subspace of (3.5), i.e.,

$$\mathcal{K} = \langle \{L_1, L_2, \ldots, L_p\} \mid \{M_1, M_2, \ldots, M_p\} \rangle.$$

For the lower bound of $\dim \mathcal{K}$, the following result is straightforwardly extended from Theorem 2.4.2.

Theorem 3.5.1 *Suppose that graph G_i is connected for each $i \in \{1, 2, \ldots, p\}$. Then it holds that*

$$\dim \mathcal{K} \geqslant \max_{\substack{k \,\in\, \{1,2,\ldots,m_i\} \\ i \,\in\, \{1,2,\ldots,p\}}} \operatorname{card}(\pi_D(\ell_k^i; G_i))$$

where $\pi_D(\ell_k^i; G_i)$ is the distance partition relative to the vertex ℓ_k^i in graph G_i.

Proof. Let $\mathcal{K}_i$ denote the controllable subspace of the network pair (L_i, M_i) for each $i \in \{1, 2, \ldots, p\}$. From Theorem 2.4.2, it follows that

$$\dim \mathcal{K}_i \geqslant \max_{k \in \{1,2,\ldots,m_i\}} \operatorname{card}(\pi_{\mathrm{D}}(\ell_k^i; G_i)).$$

Then the result is concluded from the fact that $\dim \mathcal{K} \geqslant \max_{i \in \{1,2,\ldots,p\}} \dim \mathcal{K}_i$. ∎

Next, a similar result as Theorem 2.4.4 is obtained as follows.

Theorem 3.5.2 *It holds that $\mathcal{K} \subseteq \operatorname{im} P(\pi)$ for any $\pi \in \Pi_{AEP}(\pi_L^1, \ldots, \pi_L^p)$, where π_L^i is the partition $\{\{l_1^i\}, \ldots, \{l_{m_i}^i\}, V_F^i\}$ and $P(\pi)$ denotes the characteristic matrix of the partition π.*

Proof. By definition, the subspace $\operatorname{im} P(\pi)$ is L_i-invariant and contains $\operatorname{im} M_i$ for each $i \in \{1, \ldots, p\}$ and any $\pi \in \Pi_{\mathrm{AEP}}(\pi_{\mathrm{L}}^1, \ldots, \pi_{\mathrm{L}}^p)$. Then the conclusion follows from the fact that $\mathcal{K}$ is the *smallest* L_i-invariant subspace that contains $\operatorname{im} M_i$ for each i. ∎

Theorem 3.5.2, together with (3.8), leads to the following tighter upper bound, which is similar to Theorem 2.4.8:

Theorem 3.5.3 *It holds that $\mathcal{K} \subseteq \operatorname{im} P(\pi^*_{AEP}(\pi_L^1, \ldots, \pi_L^p))$.*

The two bounds provided in Theorems 3.5.3 and 3.5.1 are tight in the sense that we can construct examples where one bound is achieved while the other holds strictly. Consider the network (3.5) with its topology switching between the two

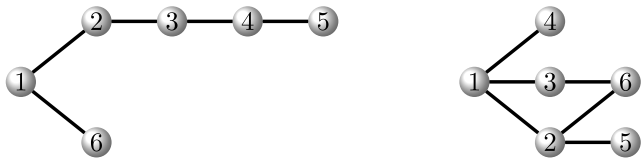

Figure 3.1: *Upper bound achieved*

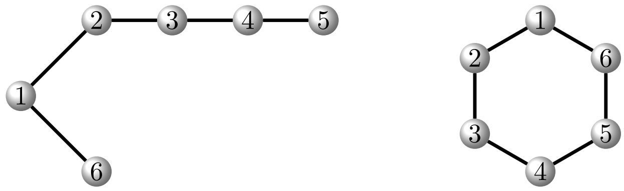

Figure 3.2: *Lower bound achieved*

graphs depicted in Figure 3.1 and the set of leaders $V_{\mathrm{L}}^i = \{1\}$ for $i = 1, 2$. It can be checked that the upper bound in Theorem 3.5.3 holds with equality and is strictly greater than the lower bound in Theorem 3.5.1. If the network (3.5) switches between the two topologies in Figure 3.2 and the leader set is still the same, the lower bound in Theorem 3.5.1 is achieved which is strictly less than the upper bound.

For the computation of the upper bound in Theorem 3.5.3, we provide an algorithm in the following. Since the algorithm provided is quite similar to the one in Theorem 2.4.10, we skip the proof.

Below, we write $\wedge\Pi'$ to denote the greatest lower bound of the set of partitions Π'.

Theorem 3.5.4 *Let ψ be the same as that in Theorem 2.4.10. Define the sequence π_k as follows*

$$\begin{aligned}\pi_0 &= \wedge\{\pi_L^1, \pi_L^2, \ldots, \pi_L^p\}\\ \pi_{k+1} &= \psi(\begin{bmatrix} P(\pi_k) & L_1 P(\pi_k) & L_2 P(\pi_k) & \cdots & L_p P(\pi_k)\end{bmatrix});\ k = 0, 1, 2, \ldots.\end{aligned}$$

Then we have

$$\pi_{k+1} \leqslant \pi_k$$

for any integer k. Moreover, there exists an integer q with $0 \leqslant q \leqslant n - m$ such that $\pi_q = \pi^*_{AEP}(\pi_L^1, \ldots, \pi_L^p) = \pi_{q+\ell}$ for all $\ell \geqslant 0$.

3.6 Concluding remarks

In this chapter, we have studied controllability of diffusively coupled networks with switching topologies. All the results can be obtained by extending relevant results in Chapter 2, which investigated controllability of networks with time-independent topologies. For controllability of networks with switching topologies, we have revealed the relationship between networks with single-integrator agents and those with general linear dynamical agents. In order to infer controllability from switching topologies, we have extended distance and almost equitable partitions to switched graphs. By using geometric control theory, we have obtained lower and upper bounds of controllable subspaces of networks with switching topologies.

Chapter 4

Partial consensus of heterogeneous diffusively coupled networks: double-integrator agents

4.1 Introduction

A network achieves consensus when all the states of the agents come to a global agreement, only through interaction with their neighbors. During the last decades, much research effort has been put into the consensus problem, see e.g. [45] and [38]. It received so much attention partly due to its wide variety of applications such as flocking of birds and groups of autonomous mobile robots.

The last decades also witnesses the effect of algebraic graph theory on the consensus problem, see e.g. [20] and [14]. When each agent has single integrator dynamics, the condition for achieving consensus has been completely highlighted in terms of graph topologies associated with networks. For instance, it has been shown that consensus is achieved for a time-varying graph topology if and only if the graph topology contains a spanning tree frequently enough when it evolves with time, see e.g. [24, 37, 46].

It is more realistic to have an insight into networks with double-integrator dynamics. For instance, some mobile robots can be feedback linearized and then described by double-integrator dynamics. The consensus problem for double-integrator networks has been of interest in the literature, see e.g. [42, 43, 44, 60, 61, 62]. For instance, it has been shown in [44] that consensus may fail for double-integrator networks even if their graph topologies contain a spanning tree. Also, some necessary and sufficient conditions have been derived for achieving consensus, see e.g. [62].

In double-integrator networks, each agent has two states that are often called *position* and *velocity*. All of the above literature assumes that the graph topology of position interaction and that of velocity interaction are identical. This assumption is not always (if sometimes) realistic. For instance, it is probable that not all velocities of mobile robots in a group can be measured due to expensive velocity sensors. Therefore, some agents can interact through both their positions and velocities with their neighbors while the other ones can only share their positions. In view of this, consensus conditions of double-integrator networks with different position and velocity graph topologies are studied in [21].

This chapter restudies this problem since the main result in [21] is not entirely correct. A counterexample will be provided in Section 4.3. As in [21], we assume that both position and velocity graphs are undirected. By taking the consensus problem as an output stability problem, we derive algebraic necessary and sufficient conditions for consensus with the help of geometric control theory, see e.g. [52]. Later, we make an effort to translate these algebraic conditions to graph theoretical conditions in terms of graph partitions.

This chapter is organized as follows. In Section 4.2, we begin with introducing the diffusively coupled networks with double-integrator dynamics as well as partial consensus problem. Then, we formulate the partial consensus problem as an output stability problem of an appropriately chosen linear system based on the dynamics of the network under investigation. In Section 4.3, we provide algebraic necessary and sufficient conditions for partial consensus problem based on the well-known solution of the output stability problem for linear systems. Also, we give a counter example to the main result of the paper [21] which studied velocity consensus for the same class of networks. Later, we give the graph theoretical interpretation of the presented algebraic necessary and sufficient conditions and provide necessary conditions in terms of almost equitable graph partitions.

4.2 Preliminaries

4.2.1 Partial consensus for double-integrator agents

Consider N agents labeled by the set $V = \{1, 2, \ldots, N\}$, each of which has double-integrator dynamics given by

$$\dot{x}_i = v_i \tag{4.1a}$$

$$\dot{v}_i = u_i \tag{4.1b}$$

where $x_i \in \mathbb{R}$ is the position of the i-th agent, $v_i \in \mathbb{R}$ is its velocity, and u_i is the diffusive coupling term defined as

$$u_i = \sum_{(i,j)\in E_x} (x_i - x_j) + \sum_{(i,j)\in E_v} (v_i - v_j) \tag{4.1c}$$

for the two undirected graphs $G_x = (V, E_x)$ and $G_v = (V, E_v)$ capturing the communication structure for positions and velocities, respectively.

Let $\tilde{G}_x = (V, \tilde{E}_x)$ and $\tilde{G}_v = (V, \tilde{E}_v)$ be two other undirected graphs. We say that the network (4.1) reaches *consensus* with respect to the pair $(\tilde{G}_x, \tilde{G}_v)$ if

$$\lim_{t\to\infty} x_i(t) - x_j(t) = 0 \quad \text{for all } (i,j) \in \tilde{G}_x, \text{ and} \tag{4.2}$$

$$\lim_{t\to\infty} v_i(t) - v_j(t) = 0 \quad \text{for all } (i,j) \in \tilde{G}_v \tag{4.3}$$

for all trajectories (x, v) of the network (4.1). In particular, we say that the network (4.1) reaches *velocity consensus* if $\tilde{E}_x = \varnothing$ and $\tilde{E}_v = V \times V$, i.e. the position communication graph is an empty graph and the velocity communication graph is the complete graph corresponding to the vertex set V.

4.2.2 Consensus as an output stability problem

In the reminder of the chapter, we will treat the consensus problem as an output stability problem. First, we recall the output stability problem for linear system in what follows.

Consider a linear system

$$\dot{x} = Ax \tag{4.4a}$$

$$y = Cx \tag{4.4b}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the output, and (C, A) is a pair of matrices with appropriate dimensions. The linear system (4.4) is *output stable* if $\lim_{t\to\infty} y(t) = 0$ for all output trajectories y.

Let p denote the characteristic polynomial of the matrix A, that is $p(\lambda) = \det(\lambda I - A)$. Also, let $p = p_- p_+$ where all roots of p_- lie in the open left half plane of $\mathbb{C}$ and those of p_+ lie in the closed right half plane of $\mathbb{C}$. Define $\mathcal{X}_-(A) = \ker p_-(A)$ and $\mathcal{X}_+(A) = \ker p_+(A)$.

The following result can be derived from [52, Ex. 4.10].

Proposition 4.2.1 *The linear system* (4.4) *is output stable if and only if* $\mathcal{X}_+(A) \subseteq \ker C$.

To formulate the consensus problem as an output stability problem, we first define $x = \mathrm{col}(x_1, x_2, \ldots, x_N)$ and $v = \mathrm{col}(v_1, v_2, \ldots, v_N)$. With these definitions, the network (4.1) can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -L_x & -L_v \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \tag{4.5a}$$

where L_x and L_v are the Laplacians of the undirected graphs $G_x = (V, E_x)$ and $G_v = (V, E_v)$, respectively. Let z be defined as

$$z = \begin{bmatrix} H_x & 0 \\ 0 & H_v \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \tag{4.5b}$$

where H_x^T and H_v^T are the incidence matrices of the graphs $\tilde{G}_x$ and $\tilde{G}_v$, respectively. Define

$$\hat{L} = \begin{bmatrix} 0 & I \\ -L_x & -L_v \end{bmatrix} \qquad \text{and} \qquad \hat{H} = \begin{bmatrix} H_x & 0 \\ 0 & H_v \end{bmatrix}. \tag{4.6}$$

With all these preparation, Proposition 4.2.1 readily implies the following result which reduces the consensus problem of the network (4.1) to the output stability problem of the linear system (4.5).

Corollary 4.2.2 *The network* (4.1) *reaches consensus with respect to the pair* $(\tilde{G}_x, \tilde{G}_y)$ *if and only if the linear system* $(\hat{H}, \hat{L})$ *is output stable, i.e.* $\mathcal{X}_+(\hat{L}) \subseteq \ker \hat{H}$.

In order to verify the condition given in Corollary 4.2.2, we have to compute two subspaces of $\mathbb{R}^{2N}$. In the next subsection, we aim at deriving another algebraic necessary and sufficient condition for consensus with respect to the pair $(\tilde{G}_x, \tilde{G}_y)$, which only requires to verify the inclusion relationships between subspaces of $\mathbb{R}^N$. To do that, we will analyze the spectrum of the matrix $\hat{L}$ by exploiting its structure.

4.3 Conditions for partial consensus

4.3.1 Algebraic necessary and sufficient conditions

Next, we present a number of auxiliary results which exploit the special structure of the matrix $\hat{L}$ in order to give a compact characterization of the subspace $\mathcal{X}_+(\hat{L})$. To do so, we first note the following well-known properties of the Laplacian L of a graph $G = (V, E)$:

$$L = L^T \geqslant 0$$
$$\operatorname{im} \mathbf{1} \subseteq \ker L$$

where $\mathbf{1}$ denote the vector whose entries are all 1.

We first prove that all eigenvalues of $\hat{L}$ lie in the closed left half plane of $\mathbb{C}$. This result is stated in the following lemma.

Lemma 4.3.1 *The real part of any eigenvalue of* $\hat{L}$ *is nonpositive.*

To prove Lemma 4.3.1, we notice the following result whose proof is straightforward.

Lemma 4.3.2 *Let* A, K, *and* Q *be* $n \times n$ *matrices with* $K = K^T \geqslant 0$, $Q = Q^T \geqslant 0$, *and* $A^T K + KA = -Q$. *If* (λ, x) *is an eigenpair of* A *where the real part of* λ *is positive, then* $x \in \ker K \cap \ker Q$.

Proof of Lemma 4.3.1. Define $\hat{K} = \begin{bmatrix} L_x & 0 \\ 0 & I \end{bmatrix}$. Note that

$$\hat{L}^T \hat{K} + \hat{K}\hat{L} = \begin{bmatrix} 0 & 0 \\ 0 & -2L_v \end{bmatrix} =: -\hat{Q} \tag{4.7}$$

and

$$\begin{aligned} \ker \hat{K} \cap \ker \hat{Q} &= (\ker L_x \times \{0\}) \cap (\mathbb{R}^n \times \ker L_v) \\ &= (\ker L_x \times \{0\}) = \ker \hat{L}. \end{aligned} \tag{4.8}$$

Suppose that $\hat{L}$ has an eigenpair (λ, ξ) where the real part of λ is positive. It follows from Lemma 4.3.2 that $\xi \in \ker \hat{K} \cap \ker \hat{Q}$. Hence, we get $\hat{L}\xi = 0$ from (4.8). This, however, implies that $\lambda = 0$. Contradiction! ■

It follows from Lemma 4.3.1 that the characteristic polynomial of $\hat{L}$, say p, can be factorized as $p(\lambda) = \lambda^k p_{\text{imag}}(\lambda) p_{\text{neg}}(\lambda)$ where k is a nonnegative integer, p_{imag} and p_{neg} are polynomials having only nonzero and purely imaginary roots and only roots with negative real part, respectively. As such, we obtain the following characterization for the subspace $\mathcal{X}_+(\hat{L})$

$$\mathcal{X}_+(\hat{L}) = \ker \hat{L}^k \oplus \ker p_{\text{imag}}(\hat{L}). \tag{4.9}$$

In order to obtain a more explicit characterization of this subspace, we utilize a number of auxiliary results.

The following lemma immediately follows from the special structure of $\hat{L}$.

Lemma 4.3.3 *The pair (λ, ξ) is an eigenpair of $\hat{L}$ if and only if $\xi = \operatorname{col}(x, \lambda x)$ for some $x \neq 0$ with $(\lambda^2 I + \lambda L_v + L_x)x = 0$.*

Next, we characterize the eigenvectors of $\hat{L}$ corresponding to purely imaginary eigenvalues.

Lemma 4.3.4 *The pair $(i\omega, \xi)$ is an eigenpair of $\hat{L}$ with $0 \neq \omega \in \mathbb{R}$ if and only if $\xi = \operatorname{col}(x, i\omega x)$ with $x \in \ker L_v$ and $L_x x = \omega^2 x$. Moreover, the algebraic multiplicity of any nonzero purely imaginary eigenvalue of $\hat{L}$ is equal to its geometric multiplicity.*

Proof. Let $(i\omega, \xi)$ be an eigenpair of $\hat{L}$ with $0 \neq \omega \in \mathbb{R}$. By pre-multiplying (4.7) by ξ^* and post-multiplying by ξ, we get $0 = 2\operatorname{Re}(i\omega)\xi^* \hat{K}\xi = -\xi^* \hat{Q}\xi$. Since $\hat{Q}$ is positive semi-definite, this yields $\hat{Q}\xi = 0$. Let $\xi = \operatorname{col}(x, v)$. Then, we get $v \in \ker L_v$. It follows from Lemma 4.3.3 that $x \in \ker L_v$ and from Lemma 4.3.4 that $(-\omega^2 I + L_x)x = 0$. This proves the first statement.

To show that the algebraic and geometric multiplicities coincide, it is sufficient to prove that the following implication holds:

$$\hat{L}\xi_1 = i\omega\xi_1 \quad \text{and} \quad \hat{L}\xi_2 = i\omega\xi_2 + \xi_1 \quad \implies \quad \xi_1 = 0. \tag{4.10}$$

Let ξ_1 and ξ_2 satisfy the left hand side of this implication. From the previous discussion, we know that $\xi = \mathrm{col}(x_1, i\omega x_1)$ for some $x_1 \in \ker L_v \cap \ker(L_x - \omega^2 I)$. By letting $\xi_2 = \mathrm{col}(x_2, v_2)$, we get

$$v_2 = i\omega x_2 + x_1 \tag{4.11}$$

$$-L_x x_2 - L_v v_2 = i\omega v_2 + i\omega x_1. \tag{4.12}$$

Note that $(\hat{L} - i\omega I)^2 \xi_2 = 0$. Since

$$(\hat{L} - i\omega I)^2 = -\begin{bmatrix} I & 0 \\ -L_v & I \end{bmatrix} \begin{bmatrix} L_x + \omega^2 I & L_v + 2i\omega I \\ \omega^2 L_v - 2i\omega L_x & L_x + \omega^2 I \end{bmatrix}, \tag{4.13}$$

we get

$$0 = (L_x + \omega^2 I)x_2 + (L_v + 2i\omega I)v_2 \tag{4.14}$$

$$0 = (\omega^2 L_v - 2i\omega L_x)x_2 + (L_x + \omega^2 I)v_2. \tag{4.15}$$

Since $x_1 \in \ker L_v$, we get

$$L_v v_2 = i\omega L_v x_2 \tag{4.16}$$

from (4.11). Then, (4.14) can be written as

$$(\omega^2 I + i\omega L_v + L_x)x_2 + 2i\omega v_2 = 0. \tag{4.17}$$

By using (4.11), we can write (4.15) as

$$-2i\omega L_x x_2 + (\omega^2 I - i\omega L_v + L_x)v_2 = 0. \tag{4.18}$$

By solving v_2 from (4.17) and substituting in (4.18), we obtain

$$[(L_x - \omega^2 I)^2 + \omega^2 L_v]x_2 = 0. \tag{4.19}$$

Further, (4.16) yields

$$(L_x - \omega^2 I)^2 x_2 = i\omega L_v v_2. \tag{4.20}$$

By multiplying (4.14) by $i\omega$ and using (4.20) and (4.11), we obtain

$$i\omega(L_x + \omega^2 I)x_2 + (L_x - \omega^2 I)^2 x_2 - 2i\omega^3 x_2 = 2\omega^2 x_1. \tag{4.21}$$

Finally, we can rewrite the last equality as

$$2\omega^2 x_1 = [L_x^2 - (2\omega^2 - i\omega)L_x + (\omega^4 - i\omega^3)I]x_2 \tag{4.22}$$
$$= (L_x - \omega^2 I)(L_x - (\omega^2 - i\omega)I)x_2. \tag{4.23}$$

Hence, we get $x_1 \in \ker(L_x - \omega^2 I) \cap \operatorname{im}(L_x - \omega^2 I)$. Note that $\ker(L_x - \omega^2 I) \cap \operatorname{im}(L_x - \omega^2 I) = \{0\}$ as $(L_x - \omega^2 I)$ is symmetric. Thus, we can conclude that $\xi_1 = \operatorname{col}(x_1, i\omega x_1) = 0$. ■

Remark 4.3.5 Theorem 1 in [21] claims that all nonzero eigenvalues of $\hat{L}$ have strictly negative real parts if $L_v \neq 0$. This is not true in general as illustrated by the following example. The network in Figure 4.1 consists of three agents. The left and right graphs indicate position and velocity interactions respectively. For this network, we have

$$L_x = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad L_v = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \tag{4.24}$$

Then it can be checked that $\lambda = \pm\sqrt{3}i$ is a purely imaginary eigenvalue of $\hat{L}$.

Figure 4.1: *Counter example*

Now, we turn our attention to the eigenvectors of $\hat{L}$ corresponding to the zero eigenvalue.

Lemma 4.3.6 *The following statements hold:*

1. $\ker \hat{L} = \ker L_x \times \{0\}$,
2. $\ker \hat{L}^2 = \ker \begin{bmatrix} L_x & L_v \\ 0 & L_x \end{bmatrix} = \ker L_x \times (\ker L_x \cap \ker L_v)$,
3. $\ker \hat{L}^3 = \ker \hat{L}^2$.

Proof. 1: Note that $\ker L_x \times \{0\} \subseteq \ker \hat{L}$. Then, it remains to prove that $\ker \hat{L} \subseteq \ker L_x \times \{0\}$. Let $\xi = \operatorname{col}(x, v) \in \ker \hat{L}$. Then,

$$0 = \hat{L}\xi = \hat{L}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ -L_x x - L_v v \end{bmatrix}.$$

Clearly, $v = 0$ and $x \in \ker L_x$. Hence, $\xi = \operatorname{col}(x, v) \in \ker L_x \times \{0\}$.

2: The first equality immediately follows from the following identity:

$$\hat{L}^2 = \begin{bmatrix} -L_x & -L_v \\ L_v L_x & -L_x + L_v^2 \end{bmatrix} = -\begin{bmatrix} I & 0 \\ -L_v & I \end{bmatrix} \begin{bmatrix} L_x & L_v \\ 0 & L_x \end{bmatrix}. \tag{4.25}$$

To show the second, it remains to prove

$$\ker \begin{bmatrix} L_x & L_v \\ 0 & L_x \end{bmatrix} \subseteq \ker L_x \times (\ker L_x \cap \ker L_v)$$

as the reverse inclusion is obvious. Let $\operatorname{col}(x, v)$ belong to the right hand side. Then, we get

$$0 = L_x x + L_v v \tag{4.26}$$

$$v \in \ker L_x \tag{4.27}$$

By pre-multiplying the latter by v^T, we get $0 = v^T L_x x + v^T L_v v = v^T L_v v$ since $v \in \ker L_x$ and L_x is symmetric. It follows from the positive semi-definiteness of L_v that $v \in \ker L_v$ and from (4.26) that $x \in \ker L_x$. Hence, we get $\operatorname{col}(x, v) \in \ker L_x \times (\ker L_x \cap \ker L_v)$.

3: Note that

$$\ker \hat{L}^3 = \hat{L}^{-1}(\ker \hat{L}^2) = \hat{L}^{-1}(\ker L_x \times (\ker L_x \cap \ker L_v)). \tag{4.28}$$

Let $\xi = \operatorname{col}(x, v) \in \ker \hat{L}^3$. Note that

$$\hat{L}\xi = \hat{L} \operatorname{col}(x, v) = \begin{bmatrix} v \\ -L_x x - L_v v \end{bmatrix} \in \ker L_x \times (\ker L_x \cap \ker L_v). \tag{4.29}$$

This means that $v \in \ker L_x$ and $L_x x + L_v v \in \ker L_x \cap \ker L_v$. The latter, however, implies that $L_x x + L_v v = 0$ as $\operatorname{im} \begin{bmatrix} L_x & L_v \end{bmatrix} \cap (\ker L_x \cap \ker L_v) = \{0\}$. Therefore, we get $\hat{L}\xi = \hat{L} \operatorname{col}(x, v) \in \ker L_x \cap \{0\} = \ker \hat{L}$. Hence, $\xi \in \ker \hat{L}^2$. This proves $\ker \hat{L}^3 \subseteq \ker \hat{L}^2$ and hence $\ker \hat{L}^3 = \ker \hat{L}^2$ as we already have $\ker \hat{L}^2 \subseteq \ker \hat{L}^3$. ■

Lemmas 4.3.4 and 4.3.6 yield that

$$\mathcal{X}_+(\hat{L}) = \ker \hat{L}^2 \oplus \big(\bigoplus_{0 \neq \omega \in \mathbb{R}, i\omega \in \sigma(\hat{L})} \ker(\hat{L} - i\omega I) \big) \tag{4.30}$$

where $\sigma(\hat{L})$ denotes the spectrum of the matrix $\hat{L}$. With all these preparations, we are ready to present the first main result of the chapter.

Theorem 4.3.7 *The network* (4.1) *reaches consensus with respect to the pair* $(\tilde{G}_x, \tilde{G}_y)$ *if and only if*

$$\begin{aligned} \ker L_x &\subseteq \ker H_x \\ \ker L_x \cap \ker L_v &\subseteq \ker H_v \\ \langle \ker L_v \mid L_x \rangle &\subseteq \ker H_x \cap \ker H_v. \end{aligned}$$

Proof. 'if': In view of Lemma 4.3.6, the first two conditions guarantee that

$$\ker \hat{L}^2 \subseteq \ker \hat{H}. \tag{4.31}$$

Let $\xi \in \ker(\hat{L} - i\omega I)$ for some $0 \neq \omega \in \mathbb{R}$ with $i\omega \in \sigma(\hat{L})$. It follows from Lemma 4.3.3 that $\xi = \mathrm{col}(x, i\omega x)$ for some x, and from Lemma 4.3.4 that $x \in \ker L_v$ and $L_x x = \omega^2 x$. Then, the subspace $\mathrm{span}\, x$ is L_x-invariant and contained in $\ker L_v$. Since $\langle \ker L_v \mid L_x \rangle$ is the largest L_x-invariant subspace that is contained in $\ker L_v$, we get $x \in \langle \ker L_v \mid L_x \rangle$. As such, the third condition yields $x \in \ker H_x \cap \ker H_v$. This implies that $\xi \in \ker \hat{H}$ since $\xi = \mathrm{col}(x, i\omega x)$. Therefore, we get

$$\ker(\hat{L} - i\omega I) \subseteq \ker \hat{H} \tag{4.32}$$

for all $0 \neq \omega \in \mathbb{R}$ with $i\omega \in \sigma(\hat{L})$. In view of (4.30), the inclusions (4.31) and (4.32) imply that

$$\mathcal{X}_+(\hat{L}) \subseteq \ker \hat{H}. \tag{4.33}$$

Then, it follows from Corollary 4.2.2 that the network (4.5) reaches consensus with respect to the pair $(\tilde{G}_x, \tilde{G}_y)$.

'only if': It follows from Corollary 4.2.2 that

$$\mathcal{X}_+(\hat{L}) \subseteq \ker \hat{H}. \tag{4.34}$$

From (4.30), we get

$$\ker \hat{L}^2 \subseteq \ker \hat{H} \tag{4.35}$$

and

$$\ker(\hat{L} - i\omega I) \subseteq \ker \hat{H} \tag{4.36}$$

for all $0 \neq \omega \in \mathbb{R}$ with $i\omega \in \sigma(\hat{L})$. The former immediately implies that the first two conditions must hold in view of Lemma 4.3.6. To conclude the proof, it remains to show that the third condition holds. Note that one can always find a basis for $\langle \ker L_v \mid L_x \rangle$ such that every basis vector is an eigenvector of L_x since $\langle \ker L_v \mid L_x \rangle$ is L_x-invariant and L_x is diagonalizable. Then, it is enough to show that the eigenvectors of L_x belonging to $\langle \ker L_v \mid L_x \rangle$ must also belong to $\ker \hat{H}$. Let $x \neq 0$

be such that $x \in \langle \ker L_v \mid L_x \rangle$ and $L_x x = \lambda x$ for some $\lambda \in \sigma(L_x)$. Suppose, first, that $\lambda = 0$. Then $x \in \ker L_x \cap \ker L_v$. Hence, we get $x \in \ker H_x \cap \ker H_v$ from the first two conditions. Now, suppose that $\lambda \neq 0$. Since L_x is positive semi-definite, $\lambda > 0$. Let ω be such that $\omega^2 = \lambda$. It follows from Lemma 4.3.3 and Lemma 4.3.4 that $\mathrm{col}(x, i\omega x) \in \ker(\hat{L} - i\omega I)$. Then, we get $x \in \ker H_x \cap \ker H_v$ from (4.36). ■

Corollary 4.3.8 *The network* (4.1) *reaches velocity consensus if and only if*

$$\ker L_x \cap \ker L_v = \mathrm{im}\,\mathbf{1} = \langle \ker L_v \mid L_x \rangle.$$

Proof. Note that velocity consensus is reached if and only if the network (4.1) reaches consensus with respect to $(\tilde{G}_x, \tilde{G}_v)$ with $\tilde{G}_x$ is the empty graph and $\tilde{G}_v$ is the complete graph. In this particular case, the necessary and sufficient conditions presented in Theorem 4.3.7 boil down to

$$\ker L_x \cap \ker L_v \subseteq \mathrm{im}\,\mathbf{1}$$
$$\langle \ker L_v \mid L_x \rangle \subseteq \mathrm{im}\,\mathbf{1}$$

since $\ker H_x = \mathbb{R}^N$ and $\ker H_v = \mathrm{im}\,\mathbf{1}$. The proof is completed by the observations that $\mathrm{im}\,\mathbf{1} \subseteq \ker L_x \cap \ker L_v$ and $\mathrm{im}\,\mathbf{1} \subseteq \langle \ker L_v \mid L_x \rangle$. ■

As mentioned before, in order to verify the conditions in Theorem 4.3.7 and Corollary 4.3.8, we only need to compute subspaces of $\mathbb{R}^N$ rather than those of $\mathbb{R}^{2N}$. Moreover, as will be seen in the next subsection, one can provide necessary graph topological conditions for the algebraic conditions of Theorem 4.3.7.

4.3.2 Graph theoretical conditions

Our next goal is to give graph theoretical interpretations of these necessary and sufficient conditions. To do so, we need to introduce some nomenclature on graph partitions.

Let $G = (V, E)$ be an undirected graph with $V = \{1, 2, \ldots, N\}$. A *cell* of G is a non-empty subset of V. We say that a collection of cells $\{C_1, C_2, \ldots, C_k\}$ is a *partition* of G if $C_i \cap C_j = \varnothing$ if $i \neq j$ and $\cup_{i=1}^{k} C_i = V$.

Let $\pi = \{C_1, C_2, \ldots, C_k\}$ be a partition. The *characteristic matrix* $P(\pi) \in \mathbb{R}^{N \times k}$ of π is defined by

$$P_{ij}(\pi) = \begin{cases} 1 & \text{if } i \in \mathcal{C}_j \\ 0 & \text{otherwise} \end{cases} \qquad 1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant k. \tag{4.37}$$

For any two partitions π_1 and π_2, we say that π_1 is *finer* than π_2, denoted by $\pi_1 \leqslant \pi_2$, if each cell of π_1 is a subset of some cell of π_2. It can be easily verified that

$$\pi_1 \leqslant \pi_2 \quad \Longleftrightarrow \quad \operatorname{im} P(\pi_2) \subseteq \operatorname{im} P(\pi_1). \tag{4.38}$$

Let Π be the set of all partitions of G. With the order "$\leqslant$", Π becomes a partially ordered set. Furthermore, it is also a complete lattice (see e.g. [11]), i.e. every subset of Π has both its greatest lower bound and the least upper bound within Π. For a subset Π' of Π, let $\wedge\Pi^{'}$ and $\vee\Pi^{'}$ denote these two bounds respectively. In particular, $\pi_1 \wedge \pi_2$ and $\pi_1 \vee \pi_2$ are used to denote $\wedge\{\pi_1, \pi_2\}$ and $\vee\{\pi_1, \pi_2\}$, respectively. Note that

$$\pi_1 \leqslant \pi_1 \vee \pi_2 \quad \text{and} \quad \pi_2 \leqslant \pi_1 \vee \pi_2 \tag{4.39}$$

$$\pi_1 \wedge \pi_2 \leqslant \pi_1 \quad \text{and} \quad \pi_1 \wedge \pi_2 \leqslant \pi_2 \tag{4.40}$$

and

$$\pi_1 \leqslant \pi \text{ and } \pi_2 \leqslant \pi \quad \Longrightarrow \quad \pi_1 \vee \pi_2 \leqslant \pi \tag{4.41}$$

$$\pi \leqslant \pi_1 \text{ and } \pi \leqslant \pi_2 \quad \Longrightarrow \quad \pi \leqslant \pi_1 \wedge \pi_2. \tag{4.42}$$

A partition $\pi = \{C_1, C_2, \ldots, C_k\}$ of G is said to be *almost equitable* if for any pair (i, j) with $1 \leqslant i \neq j \leqslant k$, there exists a number b_{ij} such that any vertex $v \in C_i$ has b_{ij} neighbors in C_j [16].

Almost equitable partitions can be characterized in terms of the invariant subspaces of the Laplacian matrix L of G as in the following lemma.

Lemma 4.3.9 *[16, Lemma 9.3.2] A partition π of G is almost equitable if and only if $\operatorname{im} P(\pi)$ is L-invariant.*

A partition $\pi = \{C_1, C_2, \ldots, C_k\}$ of G is said to be a *connectedness* partition if for any pair (i, j) with $1 \leqslant i \neq j \leqslant k$ there are no neighbors of any vertex $v \in C_i$ within C_j. Note that the trivial partition $\{V\}$ is a connectedness partition for any graph G. Also note that any connectedness partition is also almost equitable (with $b_{ij} = 0$). Let Π_C denote the set of all connectedness partitions for the graph G. For later use, we define

$$\pi_C(G) = \bigwedge \Pi_C.$$

The following lemma shows that the set Π_C is closed under the binary operation $\wedge$.

Lemma 4.3.10 *If π_1 and π_2 are two connectedness partitions for a graph G, then so is $\pi_1 \wedge \pi_2$.*

Proof. Let C and C' be two different cells of $\pi_1 \wedge \pi_2$. In view of (4.40), there exist two cells, say C_1 and C_1', of π_1 and two cells, say C_2 and C_2', of π_2 such that

$$C \subseteq C_1 \quad \text{and} \quad C \subseteq C_2 \tag{4.43}$$

$$C' \subseteq C_1' \quad \text{and} \quad C' \subseteq C_2'. \tag{4.44}$$

First, we claim that the case $C_1 = C_1'$ and $C_2 = C_2'$ is impossible. On the contrary, suppose that $C_1 = C_1'$ and $C_2 = C_2'$. Let $\pi_1 \wedge \pi_2 = \{C, C', D_1, D_2, \ldots, D_k\}$. Define $\pi_{12} = \{C \cup C', D_1, D_2, \ldots, D_k\}$. Note that

$$\pi_{12} \leqslant \pi_1 \quad \text{and} \quad \pi_{12} \leqslant \pi_2. \tag{4.45}$$

Then, we get

$$\pi_{12} \leqslant \pi_1 \wedge \pi_2 \tag{4.46}$$

from (4.42). This, however, contradicts with $C \neq C'$. As such, we proved that either $C_1 \neq C_1'$ or $C_2 \neq C_2'$. In either case, there are no vertices in the cell C which has a neighbor within C' since both π_1 and π_2 are connectedness partitions. Therefore, $\pi_1 \wedge \pi_2$ is a connectedness partition. ■

It follows from Lemma 4.3.10 that $\pi_C(G)$ is a connectedness partition. Moreover, it can be verified that

$$\ker L = \operatorname{im} P(\pi_C(G)) \tag{4.47}$$

where L is the Laplacian matrix of the graph G.

For each $\pi_0 \in \Pi$, define

$$\Pi_C(\pi_0) \triangleq \{\pi \mid \pi \in \Pi_C \text{ and } \pi_0 \leqslant \pi\}. \tag{4.48}$$

Note that the trivial partition $\{V\} \in \Pi_C(\pi_0)$ for any π_0. Therefore, $\Pi_C(\pi_0)$ is always non-empty. Let $\pi^*(\pi_0; G)$ be the greatest lower bound of the set $\Pi_C(\pi_0)$, that is

$$\pi^*(\pi_0; G) = \bigwedge \Pi_C(\pi_0). \tag{4.49}$$

Since the binary operation $\wedge$ is commutative and $\Pi_C(\pi_0)$ is a finite set, it follows from Lemma 4.3.10 that $\pi^*(\pi_0)$ is a connectedness partition of the graph G. The following lemma provides another characterization of this partition. Such a characterization will play a key role in giving a graph theoretical interpretation of the conditions of Theorem 4.3.7.

Lemma 4.3.11 $\operatorname{im} P(\pi^*(\pi_0; G)) \subseteq \langle \operatorname{im} P(\pi_0) \mid L \rangle$.

Proof. For brevity, we define $\pi^*(\pi_0) = \pi^*(\pi_0; G)$. By definition $\pi_0 \leqslant \pi^*(\pi_0)$. It follows from (4.38) that

$$\operatorname{im} P(\pi^*(\pi_0)) \subseteq \operatorname{im} P(\pi_0). \tag{4.50}$$

Since $\pi^*(\pi_0)$ is almost equitable, we know from Lemma 4.3.9 that $\operatorname{im} P(\pi^*(\pi_0))$ is L-invariant. As $\langle \operatorname{im} P(\pi_0) \mid L \rangle$ is the largest L-invariant subspace that is contained in $\operatorname{im} P(\pi_0)$, we get $\operatorname{im} P(\pi^*(\pi_0)) \subseteq \langle \operatorname{im} P(\pi_0) \mid L \rangle$. ■

With all these preparations, we can present the following necessary conditions for partial consensus. Note that the union graph is defined as $G_1 \cup G_2 = (V, E_1 \cup E_2)$ for two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$.

Theorem 4.3.12 *The network* (4.1) *reaches consensus with respect to the pair* $(\tilde{G}_x, \tilde{G}_v)$ *only if*

1. $\pi_C(\tilde{G}_x) \leqslant \pi_C(G_x)$,
2. $\pi_C(\tilde{G}_x \cup \tilde{G}_v) \leqslant \pi_C(G_x \cup G_v)$,
3. $\pi_C(\tilde{G}_x \cup \tilde{G}_v) \leqslant \pi^*(\pi_C(G_v); G_x)$.

Proof. The necessity of the first two conditions follow from the first two conditions presented in Theorem 4.3.7, (4.47), and (4.38). To prove the necessity of the last condition, first note that

$$\operatorname{im} P(\pi^*(\pi_C(G_v); G_x)) \subseteq \langle \operatorname{im} P(\pi_C(G_v)) \mid L_x \rangle \tag{4.51}$$

due to Lemma 4.3.11. Then, it follows from (4.47) that

$$\operatorname{im} P(\pi^*(\pi_C(G_v); G_x)) \subseteq \langle \ker L_v \mid L_x \rangle. \tag{4.52}$$

It follows from Theorem 4.3.7 that $\langle \ker L_v \mid L_x \rangle \subseteq \ker H_x \cap \ker H_v$ and from (4.47) $\ker H_x \cap \ker H_v = \operatorname{im} P(\pi_C(\tilde{G}_x \cup \tilde{G}_v))$. Therefore, we have

$$\operatorname{im} P(\pi^*(\pi_C(G_v); G_x)) \subseteq \operatorname{im} P(\pi_C(\tilde{G}_x \cup \tilde{G}_v)).$$

Thus, $\pi_C(\tilde{G}_x \cup \tilde{G}_v) \leqslant \pi^*(\pi_C(G_v); G_x)$ follows from (4.38). ■

In Theorem 4.3.12, we provide necessary conditions on the network topology for achieving consensus with respect to given $(\tilde{G}_x, \tilde{G}_v)$. For instance, the first condition says that a pair of agents can reach position consensus only if they can be connected by a path in graph G_x, whereas the second condition implies that a pair of agents can reach velocity consensus only if they can be connected by a path in the graph $G_x \cup G_v$. In particular, Theorem 4.3.12 provides necessary conditions for velocity consensus as follows.

Corollary 4.3.13 *The network* (4.1) *reaches velocity consensus only if*

1. $G_x \cup G_v$ *is connected,*
2. $\pi^*(\pi_C(G_v); G_x) = \{V\}$.

Proof. The velocity consensus is a special case of Theorem 4.3.12 where $\tilde{G}_x$ is the empty graph and $\tilde{G}_v$ is the complete graph. In this case, $\pi_C(\tilde{G}_x) = \{1, 2, \ldots, N\}$ and $\pi_C(\tilde{G}_v) = \pi_C(\tilde{G}_x \cap \tilde{G}_v) = \{V\}$. Then, the first condition of Theorem 4.3.12 is already satisfied and the last two become $\pi_C(G_x \cup G_v) = \{V\}$ and $\pi^*(\pi_C(G_v); G_x) = \{V\}$. Note that the former is equivalent to the union graph $G_x \cup G_v$ being connected. ■

4.4 Concluding remarks

In this chapter, we have studied the partial consensus problem for diffusively coupled networks with double-integrator (position/velocity) dynamics. In our set up, the agents share positions and velocities via (possibly) different communication networks. By looking at the partial consensus problem as an output stability problem, we derived algebraic necessary and sufficient conditions in terms of the eigenspaces of the Laplacians corresponding to position and velocity communication graphs. These algebraic conditions were translated to graph topological necessary conditions in terms of the so-called almost equitable partitions of a given graph.

Generalization of the presented results to networks with arbitrary dynamics could be a future research direction. Another line for further research is to identify classes of graphs for which the algebraic necessary and sufficient conditions can be translated to necessary and sufficient graph topological conditions.

Chapter 5

Disturbance decoupling problem of diffusively coupled networks

5.1 Introduction

Analysis and synthesis of diffusively coupled multi-agent networks have become a very popular research area in the last decade, see e.g. [24, 27, 38, 43, 47, 53]. An important issue in studying networks is to deduce dynamical properties from network topologies, which are described by the underlying graphs of the networks. For instance, it is well-known that connectivity of the underlying graphs plays a crucial role in the consensus problem, see e.g. [38]. Recently, studying other network properties from a graph theoretical perspective has attracted much attention, see e.g. [14, 19, 32, 41]. A notable instance is controllability analysis, see e.g. Chapter 2. There, graph partitions, and in particular *almost equitable partitions*, have been proven to be useful tools to analyze network controllability. Roughly speaking, almost equitable partitions can be taken as graph theoretical translations of L-invariant subspaces, with L denoting the Laplacian matrix of the underlying graph, see e.g. [16, 64].

In this chapter, we study the *disturbance decoupling problem (DDP)* of diffusively coupled networks, where each agent has single-integrator dynamics and some agents are directly affected by disturbance signals. The DDP of a network is defined in the same way as that of a linear system. For a classical linear system with inputs and outputs, the DDP amounts to finding a state feedback (if possible) such that the chosen output of the closed-loop system is not affected by disturbance signals acting on some states of the system, see e.g. [52]. If such a feedback exists, then we say the DDP for the system is solvable.

The solution of DDP for linear systems was derived from the *geometric approach*, which was inaugurated by the recognition of so-called *controlled invariant subspaces*, due independently to Basile and Marro [3] and to Wohnam and Morse [56]. Solving DDP for linear systems is in fact an immediate application of controlled invariant subspaces, see e.g. [2, 23]. Our study in this chapter inherits from but goes beyond the solution for linear systems.

We first develop *almost equitable partitions with respect to a cell*, which can be taken as generalized almost equitable partitions. This class of generalized almost equitable partitions can be taken as graph theoretical translations of a class of controlled invariant subspaces of networks. Then we establish both necessary and sufficient graph theoretical conditions for the DDP for networks. To do that, we consider both open-loop and closed-loop networks. In particular, the sufficient conditions we establish are in terms of generalized almost equitable partitions. Such graph theoretical conditions provide insight into distributed control of a network, where it is interesting and realistic to synthesize a state feedback that only requires the relative (local) information between the states of the agents rather than absolute (global) information of the states. To the authors' best knowledge, this chapter is the first attempt to study DDP for networks from a graph theoretical perspective.

The structure of this chapter is as follows. In Section 5.2, some preliminary materials are provided. Also, both the open-loop and closed-loop DDP's for diffusively coupled networks are formulated. In Section 5.3, almost equitable partitions with respect to a cell are proposed and used to characterize a class of controlled invariant subspaces. In Section 5.4, we establish graph theoretical sufficient conditions both for open-loop and closed-loop DDP's for networks. To illustrate the proposed results, a numerical example is provided in Section 5.5. The chapter ends with concluding remarks in Section 5.6.

5.2 Networks and disturbance decoupling problem

5.2.1 Diffusively coupled networks with disturbance

In this chapter, we consider a multi-agent network consisting of $n > 1$ agents labeled by the set $V = \{1, 2, \dots, n\}$. We assign three subsets of V as follows: $V_{\mathrm{L}} = \{\ell_1, \ell_2, \dots, \ell_m\}$ where $m \leqslant n$, $V_{\mathrm{F}} = V \setminus V_{\mathrm{L}}$ and $V_{\mathrm{D}} = \{w_1, w_2, \dots, w_r\}$ where $r \leqslant n$.

We associate the dynamics

$$\dot{x}_i(t) = \begin{cases} z_i(t) + u_k(t) + d_l(t) & \text{if } i = w_l \in V_{\mathrm{D}} \\ z_i(t) + u_k(t) & \text{otherwise} \end{cases} \tag{5.1a}$$

to each agent $i = \ell_k \in V_{\mathrm{L}}$, and

$$\dot{x}_i(t) = \begin{cases} z_i(t) + d_l(t) & \text{if } i = w_l \in V_{\mathrm{D}} \\ z_i(t) & \text{otherwise} \end{cases} \tag{5.1b}$$

to each agent $i \in V_{\mathrm{F}}$, where $x_i \in \mathbb{R}$ represents the state of agent $i \in V$, z_i indicates the coupling variable of agent $i \in V$, $u_k \in \mathbb{R}$ is an external control input signal received by agent $i = \ell_k \in V_{\rm L}$, and $d_l \in \mathbb{R}$ is taken as an external disturbance signal influencing agent $i = w_l \in V_{\rm D}$.

Considering the roles of the defined subsets of V, we refer to $V_{\rm L}$ as the *leader set*, V_F as the *follower set*, and V_D as the *disturbance set*. Correspondingly, we say i is a *leader* if $i \in V_L$, and i is a *follower* if $i \in V_F$.

We consider a simple (unweighted) directed graph $G = (V, E)$, where V is the vertex set and $E \subseteq V \times V$ is the arc set of G. For two distinct vertices $i, j \in V$, we have $(i, j) \in E$ if there is an arc from i to j with i being the *tail* and j being the *head* of the arc. Then i is said to be a *neighbor* of j. The coupling variable z_i admits the following *diffusive coupling* rule:

$$z_i(t) = -\sum_{(j,i)\in E} (x_i(t) - x_j(t)). \tag{5.1c}$$

By defining $x(t) = \mathrm{col}(x_1(t), x_2(t), \cdots, x_n(t))$, $u(t) = \mathrm{col}(u_1(t), u_2(t), \cdots, u_m(t))$ and $d(t) = \mathrm{col}(d_1(t), d_2(t), \cdots, d_r(t))$, we write network (5.1) into a compact form as follows:

$$\dot{x}(t) = -Lx(t) + Mu(t) + Sd(t) \tag{5.2}$$

where L is the in-degree Laplacian of the simple directed graph G (see e.g. [32, p.26]), the matrix $M \in \mathbb{R}^{n\times m}$ is defined by

$$M_{ik} = \begin{cases} 1 & \text{if } i = \ell_k \\ 0 & \text{otherwise} \end{cases}$$

and the matrix $S \in \mathbb{R}^{n\times r}$ is defined by

$$S_{il} = \begin{cases} 1 & \text{if } i = w_l \\ 0 & \text{otherwise.} \end{cases}$$

Next we consider another simple directed graph $\tilde{G} = (V, \tilde{E})$ and define the output $y(t)$ of the network (5.2) as follows:

$$y(t) = R^\top x(t) \tag{5.3}$$

where R is the *incidence matrix* of $\tilde{G}$ [32, p.23]. The output variables (5.3) capture the differences between the state components of certain pairs of agents determined by the arc set $\tilde{E}$ of $\tilde{G}$. In particular, an arc from i to j in $\tilde{G}$ corresponds to the output variable $x_i - x_j$ in (5.3).

In this chapter, we study the so-called *disturbance decoupling problem* (DDP) for network (5.2) by establishing graph topological conditions. Roughly speaking, our aim is to investigate the effect of the disturbance signal d on the output y, given by (5.3). For a formal description of the problem and discussing the proposed results, we first review the DDP and its solution for general linear systems.

5.2.2 Review: disturbance decoupling problem for linear systems

Consider the linear system

$$\dot{x}(t) = Ax(t) + Ed(t) \tag{5.4a}$$

$$y(t) = Cx(t) \tag{5.4b}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^r$ is the disturbance, $y \in \mathbb{R}^q$ is the output, and all matrices involved are of appropriate dimensions. We denote the state trajectory of the system (5.4) for the initial state $x(0) = x_0$ and the disturbance d by $x^{x_0,d}$ and the corresponding output trajectory by $y^{x_0,d}$.

We say that system (5.4) is *disturbance decoupled* if $y^{x_0,d_1}(t) = y^{x_0,d_2}(t)$ for all $x_0 \in \mathbb{R}^n$, all locally-integrable disturbances d_1, d_2, and all $t \in \mathbb{R}$. Due to linearity, this is equivalent to the condition $y^{0,d_1}(t) = y^{0,d_2}(t)$ for all locally-integrable disturbances d_1, d_2, and all $t \in \mathbb{R}$.

In what follows, we quickly review the geometric approach for the DDP. For more details, we refer to [55] and [52].

Let $\langle A \mid \operatorname{im} E\rangle$ denote the controllable subspace corresponding to the matrix pair (A, E), i.e., $\langle A \mid \operatorname{im} E\rangle = \operatorname{im} E + A \operatorname{im} E + \cdots + A^{n-1} \operatorname{im} E$. As is well-known, the subspace $\langle A \mid \operatorname{im} E\rangle$ is the smallest A-invariant subspace that contains $\operatorname{im} E$. Note that we call a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ *A-invariant* if $A\mathcal{V} \subseteq \mathcal{V}$ where $A : \mathbb{R}^n \to \mathbb{R}^n$. For the matrix pair (A, C), the unobservable subspace is denoted by $\langle \ker C \mid A\rangle$, i.e., $\langle \ker C \mid A\rangle = \ker C \cap A^{-1} \ker C \cap \cdots \cap A^{-n+1} \ker C$. Here, for a given subspace $\mathcal{X}$, $A^{-1}\mathcal{X}$ denotes the subspace $\{x : Ax \in \mathcal{X}\}$. It is well-known that the unobservable subspace $\langle \ker C \mid A\rangle$ is the largest A-invariant subspace that is contained in $\ker C$.

Necessary and sufficient conditions for the system (5.4) to be disturbance decoupled is well-known and are recapped in the following lemma.

Lemma 5.2.1 *The following conditions are equivalent.*

1. *System* (5.4) *is disturbance decoupled.*
2. *There exists an A-invariant subspace $\mathcal{V}$ such that* $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker C$.
3. *The inclusion* $\operatorname{im} E \subseteq \langle \ker C \mid A\rangle$ *holds.*
4. *The inclusion* $\langle A \mid \operatorname{im} E\rangle \subseteq \ker C$ *holds*

Note that the equivalence between the first three statements is quite standard and can be found in [52, Ch. 4]. The fourth statement immediately follows from the first two and will be employed in the context of networks later.

Now, suppose that the linear system (5.4) is not disturbance decoupled. Then, one may think of applying control inputs to manipulate the system dynamics such that the closed-loop system will be disturbance decoupled. To do that, consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \tag{5.5a}$$
$$y(t) = Cx(t) \tag{5.5b}$$

where $u \in \mathbb{R}^m$ is the input and $B \in \mathbb{R}^{n\times m}$. The *disturbance decoupling problem* (DDP) by state feedback for the system (5.5) amounts to finding a state feedback of the form $u = Kx$ such that the resulting closed-loop system

$$\dot{x}(t) = (A + BK)x(t) + Ed(t) \tag{5.6a}$$
$$y(t) = Cx(t) \tag{5.6b}$$

is disturbance decoupled. Moreover, if such a state feedback exists, then we say the DDP for system (5.5) is *solvable*.

Necessary and sufficient conditions for solvability of the DDP are among the classical results of the geometric approach. In order to state these classical results, we need to review a few more notions from the geometric approach. We say that a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is *controlled invariant* for the pair (A, B) if there exists a matrix K such that $(A + BK)\mathcal{V} \subseteq \mathcal{V}$. Moreover, we have

$$\mathcal{V} \text{ is controlled invariant for } (A, B) \Leftrightarrow A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B.$$

For the pair (A, B), we denote the set of all controlled invariant subspaces which are contained in $\ker C$ by $\mathscr{V}(A, B, C)$. Let $\mathcal{V}^*(A, B, C)$ denote the maximal element of the set $\mathscr{V}(A, B, C)$ with respect to the partial order induced by the subspace inclusion, that is $\mathcal{V} \subseteq \mathcal{V}^*(A, B, C)$ for all $\mathcal{V} \in \mathscr{V}(A, B, C)$. The existence and uniqueness of such an element immediately follow from finite-dimensionality. It is well-known that $\mathcal{V}^*(A, B, C) \in \mathscr{V}(A, B, C)$. Now, the following lemma states necessary and sufficient conditions for the solvability of the disturbance decoupling problem for system (5.5).

Lemma 5.2.2 *Considering system* (5.5)*, the following statements are equivalent:*

1. *The disturbance decoupling problem for system* (5.5) *is solvable.*

2. *There exists a controlled invariant subspace $\mathcal{V}$ for the pair (A, B) such that* $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker C$.

3. *The inclusion* $\operatorname{im} E \subseteq \mathcal{V}^*(A, B, C)$ *holds.*

5.2.3 Disturbance decoupling problem of networks

In this subsection, we formally state the DDP for diffusively coupled networks. Recall the network (5.2) together with the output (5.3). Similar to Subsection 5.2.2, we first consider the open-loop case where no external control input is applied to the agents. In such a case, we consider the network and its output given by

$$\dot{x}(t) = -Lx(t) + Sd(t) \tag{5.7a}$$

$$y(t) = R^\top x(t) \tag{5.7b}$$

where the matrices L, S, and R are defined as before. We are interested in investigating graph theoretical conditions under which the network (5.7) is disturbance decoupled. Such conditions will be given in Subsection 5.4.1.

In case the network (5.7) is not disturbance decoupled, similar to the idea in Subsection 5.2.2, we are interested in rendering the network to be disturbance decoupled by choosing some agents as leaders and apply appropriate inputs to these agents. In such a case, we consider the network and its output given by

$$\dot{x}(t) = -Lx(t) + Mu(t) + Sd(t) \tag{5.8a}$$

$$y(t) = R^\top x(t) \tag{5.8b}$$

where the matrices L, M, S, and R are defined as before. For the network (5.8), we study graph theoretical conditions under which the disturbance decoupling problem is solvable. Such conditions will be presented in Subsection 5.4.2.

5.3 Graph partitions: extension for disturbance decoupling problem

To provide graph theoretical conditions for DDP of networks, we employ and develop graph partitions. In this section, we first review some notions of graph partitions in simple directed graphs. Recall that the graph partitions were employed *only* for simple undirected graphs in previous chapters. Then, we study in details so-called *almost equitable partitions with respect to a cell* to provide graph theoretical solutions of DDP.

Let $G = (V, E)$ be a simple (unweighted) directed graph where $V = \{1, 2, \ldots, n\}$, $E \subseteq V \times V$, and $(i, i) \notin E$. By $L(G)$, we denote the in-degree Laplacian of G [32, p. 26]. We simply use L to denote the Laplacian matrix when the underlying graph is clear from the context.

We call any subset of V a *cell* of V. We call a collection of cells, given by $\rho = \{C_1, C_2, \dots, C_k\}$, a *partial partition* of V if $C_i \cap C_j = \varnothing$ whenever $i \neq j$. In addition, we call ρ a *partition* of V if it is a partial partition and $\cup_i C_i = V$. At some occasions, to clarify the underlying graph we say ρ is a (partial) partition of $G = (V, E)$, or shortly G, meaning that ρ is a (partial) partition of V.

For a cell $C \subseteq V$, we define the *characteristic vector* of C as

$$p_i(C) = \begin{cases} 1 & \text{if } i \in C \\ 0 & \text{otherwise.} \end{cases}$$

For a (partial) partition $\rho = \{C_1, C_2, \dots, C_k\}$, we define the *characteristic matrix* of ρ as

$$P(\rho) = \begin{bmatrix} p(C_1) & p(C_2) & \cdots & p(C_k) \end{bmatrix}.$$

Finally, the notion of partial ordering for partitions is defined as follows. We say that a partition π_1 is *finer* than another partition π_2, or alternatively π_2 is *coarser* than π_1, if each cell of π_1 is a subset of some cell of π_2 and we write $\pi_1 \leqslant \pi_2$. Also we write as $\pi_1 \nleqslant \pi_2$ meaning that π_1 is not finer than π_2. It is a direct consequence of the definition that

$$\pi_1 \leqslant \pi_2 \quad \Longleftrightarrow \quad \operatorname{im} P(\pi_2) \subseteq \operatorname{im} P(\pi_1).$$

5.3.1 Almost equitable partitions in directed graphs

Here, we adopt the notion of almost equitability (see e.g. [16]) for directed graphs. For a given cell $C \subseteq V$, we write

$$N(j, C) = \{i \in C : (i, j) \in E\}.$$

We call a partition $\pi = \{C_1, C_2, \dots, C_k\}$ an *almost equitable partition* (AEP) of G if for each $i, j \in \{1, 2, \cdots, k\}$ with $i \neq j$ there exists an integer d_{ij} such that $|N(v, C_j)| = d_{ij}$ for all $v \in C_i$.

Example 5.3.1 Consider the graph H depicted in Figure 5.1. It is easy to verify that the partition π given by

$$\pi = \{\{1, 2\}, \{7, 8\}, \{4, 6\}, \{3\}, \{5\}\} \tag{5.9}$$

is an almost equitable partition of H.

For a given matrix A, we denote its (i, j)-th element by A_{ij}. Then, associated to an almost equitable partition $\pi = \{C_1, C_2, \dots, C_k\}$, we define the matrix $\mathcal{L}_\pi$ as:

$$(\mathcal{L}_\pi)_{ij} = \begin{cases} -d_{ij} & \text{if } i \neq j \\ s_i & \text{otherwise} \end{cases} \tag{5.10}$$

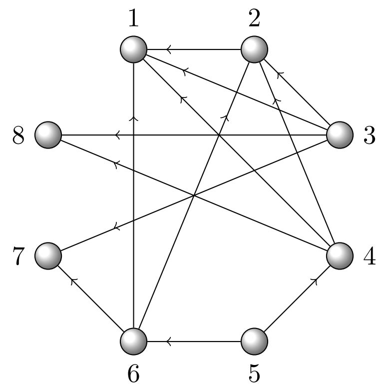

Figure 5.1: *A simple directed graph H*

where $s_i = \sum_{j \neq i} d_{ij}$.

For undirected graphs, characterization of almost equitable partitions in terms of invariant subspaces has been provided in [63]. In particular, it is shown that a partition is almost equitable if and only if the image of its characteristic matrix is L-invariant. This result can be extended to the case of directed graphs as stated in the following lemma.

Lemma 5.3.2 *A partition $\pi = \{C_1, C_2, \ldots, C_k\}$ is an AEP of G if and only if $\operatorname{im} P(\pi)$ is L-invariant.*

Proof. First, we prove the "only if" part. Assume that π is an AEP of G, and let $\mathcal{L}_\pi$ be defined as in (5.10). We claim that

$$LP(\pi) = P(\pi)\mathcal{L}_\pi, \tag{5.11}$$

and, hence $\operatorname{im} P(\pi)$ is L-invariant. First, we show that

$$(LP(\pi))_{rj} = (P(\pi)\mathcal{L}_\pi)_{rj} \tag{5.12}$$

for $r = \{1, 2, \ldots, n\}$, $j = \{1, 2, \ldots, k\}$, and $r \notin C_j$. Clearly, the left hand side is equal to $-|N(r, C_j)|$. Now, since π is an AEP, we have $-|N(r, C_j)| = -d_{ij}$ where i is such that $r \in C_i$. The right hand side of (5.12) is equal to $(\mathcal{L}_\pi)_{ij}$ which is again equal to $-d_{ij}$ by definition. Hence, it remains to show that the equality (5.12) also holds for the remaining n elements indicated by $r \in C_j$. To show this, obviously, it suffices to prove that the row sums of the matrix $LP(\pi)$ is equal to that of $P(\pi)\mathcal{L}_\pi$. Let $\mathbb{1}_k$ denote the vector of ones with the length k. Then by multiplying the left hand side of

(5.11) by $\mathbb{1}_k$, we obtain that $LP(\pi)\mathbb{1}_k = L\mathbb{1}_n = 0$. Similarly, for the right hand side we have $P(\pi)\mathcal{L}_\pi\mathbb{1}_k = 0$ as the row sums of $\mathcal{L}_\pi$ is zero. Therefore, (5.11) holds, and thus $\operatorname{im} P(\pi)$ is L-invariant.

Conversely, assume that $\operatorname{im} P(\pi)$ is L-invariant. Then, for each $j = 1, 2, \ldots, k$, we have

$$Lp(C_j) \in \operatorname{im} P(\pi) \tag{5.13}$$

as $p(C_j)$ is contained in the image of $P(\pi)$. Observe that the q^{th} element of $Lp(C_j)$ is equal to $|N(q, C_j)|$ for each $q \notin C_j$. Hence, based on (5.13), for any $q_1, q_2 \in C_i$, we have $|N(q_1, C_j)| = |N(q_2, C_j)|$ for all $j \neq i$. Consequently, π is an AEP of G. ■

Note that, if π is an AEP, then based on the proof of Lemma 5.3.2 we have

$$LP(\pi) = P(\pi)X \tag{5.14}$$

for $X = \mathcal{L}_\pi$ where $\mathcal{L}_\pi$ is given by (5.10). Moreover, $\mathcal{L}_\pi$ is the unique solution of (5.14) as $P(\pi)$ has full column rank.

5.3.2 Almost equitable partitions with respect to a cell

Given a cell C and a partition $\pi = \{C_1, C_2, \ldots, C_k\}$ in a directed graph G, we call π an *AEP with respect to* C if for each $i, j \in \{1, 2, \cdots, k\}$ with $i \neq j$ there exists an integer d_{ij} such that $|N(v, c_j)| = d_{ij}$ for all $v \in C_i \backslash C$.

Observe that if π is an AEP, then the number of neighbors that a vertex in C_i has in C_j is independent of the choice of the vertex in C_i, for all $i, j \in \{1, 2, \cdots, k\}$ with $i \neq j$. The notion of almost equatability with respect to a cell C is obtained by exempting the nodes in C from satisfying neighborhood constraints of the ordinary almost equitability. Clearly, π is an AEP of G if and only if it is an AEP with respect to the empty cell. Moreover, if π is an AEP of G, then it is an AEP with respect to any arbitrary cell of V.

Example 5.3.3 Consider the graph G given by Figure 5.2, and let the partition π be given by (5.9). Then it is easy to observe that π is an almost equitable partition of G with respect to $C = \{2\}$.

Note that if π is an almost equitable partition with respect to C, then, similar to ordinary almost equitable partitions, we can define the matrix $\mathcal{L}_\pi$ as in (5.10). In this case we use the notation $\mathcal{L}_\pi^C$ to distinguish from the case of ordinary almost equitability.

Our aim, here, is to characterize the property of almost equitability with respect to a cell. To state this characterization, we need some additional notation and auxiliary results.

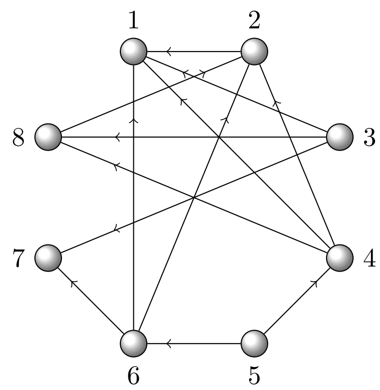

Figure 5.2: *A simple directed graph G*

For a given matrix $M \in \mathbb{R}^{m \times m}$, let M^α with $\alpha \subseteq \{1, 2, \cdots, m\}$ denote the submatrix of M obtained by collecting the rows of M indexed by α. Then, the following result holds.

Lemma 5.3.4 *A partition π is an AEP with respect to cell C if and only if*

$$L^\alpha \operatorname{im} P(\pi) \subseteq \operatorname{im} P^\alpha(\pi) \tag{5.15}$$

where $\alpha = V \backslash C$.

Proof. The proof is analogous to that of Lemma 5.3.2 by restricting the rows of L and P to those which are indexed by α. ■

Note that, if π is an AEP with respect to C, we have

$$L^\alpha P(\pi) = P^\alpha(\pi) \mathcal{L}_\pi^C \tag{5.16}$$

where $\alpha = V \backslash C$, and $\mathcal{L}_\pi^C$ is given by (5.10) with d_{ij}s obtained from the definition of almost equitability with respect to C. Now, we have the following characterization for almost equitability with respect to a cell.

Theorem 5.3.5 *Let $C = \{\ell_1, \ell_2, \ldots, \ell_m\}$ be a cell of V and $\pi = \{C_1, C_2, \ldots, C_k\}$ be a partition of G. Then the following statements are equivalent:*

1. *The partition π is an AEP of G with respect to C.*

2. *$L \operatorname{im} P(\pi) \subseteq \operatorname{im} P(\pi) + \operatorname{im} P(\rho)$ where $\rho = \{\{\ell_1\}, \{\ell_2\}, \ldots, \{\ell_m\}\}$.*

3. *There exists a simple (unweighted) directed graph $H = (V, F)$ obtained from $G = (V, E)$ by adding some non-existing or removing some existing arcs from a vertex in V to a vertex in C such that π is an almost equitable partition of H.*

Proof. First we show that the first two statements are equivalent. It is easy to observe that the second statement is equivalent to:

$$\begin{bmatrix} L_{\bar{\alpha}} \\ L^{\alpha} \end{bmatrix} \operatorname{im} P(\pi) \subseteq \operatorname{im} \begin{bmatrix} P^{\bar{\alpha}}(\pi) \\ P^{\alpha}(\pi) \end{bmatrix} + \operatorname{im} \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \tag{5.17}$$

where $\alpha = V \backslash C$ and $\bar{\alpha} = C$. This holds if and only if

$$L^{\alpha} \operatorname{im} P(\pi) \subseteq \operatorname{im} P^{\alpha}(\pi)$$

which is equivalent to almost equitability of π with respect to C by Lemma 5.3.4.

Now, by assuming that the first two statements hold, we prove the third statement as follows. Since π is an AEP with respect to C, the equality (5.16) holds. Let the matrices X and Y be defined as $X = \mathcal{L}_{\pi}^{C}$ and $Y = L^{\bar{\alpha}} P(\pi) - P^{\bar{\alpha}}(\pi) \mathcal{L}_{\pi}^{C}$. Then, clearly, we have

$$\begin{bmatrix} L^{\bar{\alpha}} \\ L^{\alpha} \end{bmatrix} P(\pi) = \begin{bmatrix} P^{\bar{\alpha}}(\pi) \\ P^{\alpha}(\pi) \end{bmatrix} X + \begin{bmatrix} I_m \\ 0 \end{bmatrix} Y.$$

Now, for each $i = \{1, 2, \cdots, m\}$, let r_i be an integer such that $\ell_i \in C_{r_i}$. Then, it is easy to observe that the matrix Y is obtained as

$$Y_{ij} = -|N(\ell_i, C_j)| - (\mathcal{L}_{\pi}^{C})_{r_i j} \tag{5.18}$$

for each $i \in \{1, 2, \ldots, m\}$, $j = \{1, 2, \ldots, k\}$, and $j \neq r_i$. The remaining m elements of Y are such that $Y \mathbb{1} = 0$. By (5.10), the equality (5.18) can be rewritten as

$$Y_{ij} = -|N(\ell_i, C_j)| + d_{r_i j} \tag{5.19}$$

where $d_{r_{ij}}$ are obtained from the definition of almost equitability with respect to C.

Now, we construct the graph $H = (V, F)$ by adding some non-existing arcs or removing some existing arcs of G as follows. For each $i \in \{1, 2, \cdots, m\}$ and $j = \{1, 2, \cdots, k\}$, we add a total number of Y_{ij} arcs from some available nodes in C_j to ℓ_i if $Y_{ij} > 0$. Note that multiple arcs between two vertices is not allowed. This is always possible since $d_{r_i j} \leqslant |C_j|$, and hence $Y_{ij} \leqslant |C_j| - |N(\ell_i, C_j)|$. Similarly, if $Y_{ij} < 0$, we remove a total number of $|Y_{ij}|$ existing arcs which are from some nodes in C_j to ℓ_i. This is also always implementable, as $-Y_{ij} \leqslant |N(\ell, C_j)|$. Denoting the arc set obtained in this way by F, it is easy to observe that the partition π is an AEP of $H = (V, F)$ by construction.

It remains to show that the third statement implies either of the other two. Assume that there exists a simple graph $H = (V, F)$ obtained from $G = (V, E)$ by adding some non-existing or removing some existing arcs from some vertices in V to vertices in C such that π is an almost equitable partition of H. Let $L(H)$ denote the Laplacian matrix of H. Then, by Lemma 5.3.2, we have

$$L(H)P(\pi) = P(\pi)X \tag{5.20}$$

for some matrix X. Hence, $L^\alpha(H)P(\pi) = P^\alpha(\pi)X$ for $\alpha = V \backslash C$. Now, since the head of all arcs which are added or removed from G are all in C, we have $L^\alpha(H) = L^\alpha(G)$. Consequently, π is an AEP of G with respect to C by Lemma 5.3.4. ■

Example 5.3.6 As mentioned in Example 5.3.3, the partition π given by (5.9) is an almost equitable partition of G with respect to C, where $C = \{2\}$ and the graph G is given by Figure 5.2. Moreover, as pointed out in Example 5.3.1, π is an AEP of the graph H given by Figure 5.2. Now, observe that graph H is obtained from G by removing the arc from vertex 8 to 2, and adding an arc from vertex 3 to 2. This indeed corresponds to the equivalence between the first and the third statement of Theorem 5.3.5.

5.4 Graph theoretical conditions for the disturbance decoupling problem

Recall that $V_D = \{w_1, w_2, \ldots, w_r\}$. We assume without loss of generality that leaders are not affected by the disturbance signals, i.e. $V_L \cap V_D = \varnothing$. Indeed, it is easy to show that if $V_L \cap V_D$ is nonempty then one can redefine the leader set as $V_L' = V_L \backslash (V_L \cap V_D)$, and solve the DDP with respect to the leader set V_L'. Now, let the partition π_S of V be defined as

$$\pi_S = \{\{w_1\}, \{w_2\}, \ldots, \{w_r\}, V \setminus V_D\}. \tag{5.21}$$

Obviously, we have

$$\operatorname{im} S \subseteq \operatorname{im} P(\pi_S). \tag{5.22}$$

Moreover, it is easy to observe that there exists a partition of G, say π_R such that

$$\operatorname{im} P(\pi_R) = \ker R^\top. \tag{5.23}$$

Then, the following result holds.

Lemma 5.4.1 *The network* (5.7) *is disturbance decoupled only if $\pi_R \leqslant \pi_S$. Similarly, the disturbance decoupling problem for the network* (5.8) *is solvable only if $\pi_R \leqslant \pi_S$ holds.*

Proof. Suppose that the network (5.7) is disturbance decoupled, or the DDP for the network (5.8) is solvable. Then it follows from Lemmas 5.2.1 and 5.2.2 that $\operatorname{im} S \subseteq \ker R^\top = \operatorname{im} P(\pi_R)$. Hence, by considering (5.21) and the structure of S, we conclude that $\pi_R \leqslant \pi_S$. ∎

Next, we provide graph theoretical sufficient conditions for the open-loop and the closed-loop DDP, respectively.

5.4.1 Open-loop cases

The following theorem gives a sufficient graph theoretical condition for the network (5.7) to be disturbance decoupled.

Theorem 5.4.2 *Let π_S and π_R be given by* (5.21) *and* (5.23)*, respectively. Then the network* (5.7) *is disturbance decoupled if there exists a partition π such that both of the following conditions hold*

1. *π is an AEP of G*
2. *$\pi_R \leqslant \pi \leqslant \pi_S$*

Proof. Suppose that both conditions hold. Then, by Lemma 5.3.2, we obtain that $\operatorname{im} P(\pi)$ is L-invariant and $\operatorname{im} S \subseteq \operatorname{im} P(\pi) \subseteq \ker R^\top$. Hence, it follows from Lemma 5.2.1 that (5.7) is disturbance decoupled. ∎

Remark 5.4.3 The second condition of Theorem 5.4.2 can be checked as follows. Given the partition π_S, consider the set

$$\Pi_{\text{AEP}}(\pi_S) = \{\pi \text{ is almost equitable } \mid \pi \leqslant \pi_S\}. \tag{5.24}$$

One can employ the algorithm given in Theorem 2.4.10 to find the least upper bound $\pi^*_{\text{AEP}}(\pi_S)$ of the set (5.24). Then one can easily verify whether the partial order relationship $\pi_R \leqslant \pi^*_{\text{AEP}}(\pi_S)$ holds.

5.4.2 Closed-loop cases

In case DDP is not solvable for the network (5.7), one may try to make (5.7) disturbance decoupled by using a control input.

Recall the notion of controlled invariant subspaces in Subsection 5.2.2. As we are dealing with graph topological conditions, we are not interested in all subspaces but only those which can be written as an image of a partition. As observed in the previous subsection, almost equitable partitions correspond to L-invariant subspaces. Now, the following Lemma establishes the relationship between almost equitable partitions with respect to a cell and controlled invariance of the pair (L, M).

Lemma 5.4.4 *For a given graph G, let V_L, M, and L be defined as before. Let π be a partition of G. Then $\operatorname{im} P(\pi)$ is controlled invariant for the pair (L, M) if and only if π is an almost equitable partition with respect to V_L.*

Proof. Note that $\operatorname{im} M = \operatorname{im} P(\rho)$ where $\rho = \{\{\ell_1\}, \{\ell_2\}, \ldots, \{\ell_m\}\}$. The claim follows immediately from Theorem 5.3.5. ■

Now we are in the position to apply the results of Section 5.3.2 to DDP of the network (5.8). This is discussed in the following theorem.

Theorem 5.4.5 *Let V_L, π_R, and π_S be defined as before. Then the DDP for the network* (5.8) *is solvable if there exists a partition π of G such that both of the following conditions hold*

1. *π is almost equitable with respect to V_L*
2. *$\pi_R \leqslant \pi \leqslant \pi_S$*

Proof. Suppose that both conditions hold. Then, by Lemma 5.4.4, $\operatorname{im} P(\pi)$ is controlled invariant for the pair (L, M). Moreover, we have $\operatorname{im} P(\pi_S) \subseteq \operatorname{im} P(\pi) \subseteq \operatorname{im} P(\pi_R)$. Hence, by (5.22) and (5.23), we obtain that $\operatorname{im} S \subseteq \operatorname{im} P(\pi) \subseteq \ker R^\top$. Consequently, the DDP for the network (5.8) is solvable by Lemma 5.2.2. ■

Suppose that the conditions of Theorem 5.4.5 hold, and hence the DDP for the network (5.8) is solvable. This means that there exists a state feedback $u(t) = Kx(t)$ such that the resulting closed-loop network is disturbance decoupled. To see the structure of the controller, note that π is an AEP of $G = (V, E)$ with respect to the leader set V_L. Hence, π is an AEP of $H = (V, F)$ where H is obtained from G by adding or removing arcs from vertices in V to vertices in V_L, as discussed in the proof of Theorem 5.3.5. These adding and removing of the arcs are indeed associated with the state feedback controller which makes the network (5.8) disturbance decoupled. In particular, it is easy to observe that adding an arc from a vertex in V, say i, to a vertex in V_L, say ℓ_k, corresponds to the control signal $x_i(t) - x_{\ell_k}(t)$ which is to be applied to the leader vertex ℓ_k. Similarly, removing an arc from i to ℓ_k corresponds to the term $x_{\ell_k}(t) - x_i(t)$ in the control signal. Consequently, the controller can be expressed as

$$u_k(t) = \sum_{(j,\ell_k)\in F\setminus E} (x_j(t) - x_{\ell_k}(t)) - \sum_{(j,\ell_k)\in E\setminus F} (x_j(t) - x_{\ell_k}(t)) \tag{5.25}$$

for each $k = \{1, 2, \ldots, m\}$. This shows that, as demanded in the context of distributed control, the controller only uses the relative information of the states of the agents to achieve disturbance decoupling for network (5.8). Observe that, by applying the controller (5.25) to network (5.8), we obtain the following input/state/output network:

$$\dot{x}(t) = -L(H)x(t) + Sd(t) \tag{5.26a}$$

$$y(t) = R^\top x(t), \tag{5.26b}$$

where $L(H)$ denotes the Laplacian matrix of the graph $H = (V, F)$. The network (5.26) is indeed disturbance decoupled by Theorem 5.4.2, as π is an AEP of H and $\pi_R \leqslant \pi \leqslant \pi_S$. This is in accordance with the fact that the DDP for network (5.8) is solvable.

Remark 5.4.6 The sufficient conditions in Theorem 5.4.5 are more involved to check than those in Theorem 5.4.2. For given π_S and V_{L}, we consider the set

$$\Pi_{\mathrm{GAEP}}(\pi_S) = \{\pi \text{ is almost equitable with respect to } V_{\mathrm{L}} \mid \pi \leqslant \pi_S\}. \tag{5.27}$$

In general, we cannot guarantee that the least upper bound of the set (5.27) still belongs to this set. As such, conditions of Theorem 5.4.2 cannot be verified in the same way as those in Theorem 5.4.2.

Remark 5.4.7 It is worth mentioning that, in general, one should not expect necessary and sufficient conditions in terms of graph partitions either for the network (5.7) to be disturbance decoupled or for the DDP of (5.8) to be solvable. The reason is that not all the subspaces can be written in terms of the image of the characteristic matrix of a partition. In fact, the lack of necessary and sufficient conditions here, are mainly associated with the gap between the image of the characteristic matrices of partitions and arbitrary subspaces.

5.5 A numerical example

To illustrate the proposed results, consider the network (5.2) with the communication graph G as shown in Figure 5.3 (left). For this network, let black vertices denote the leaders, i.e. $V_{\mathrm{L}} = \{2\}$. Also let the square vertices correspond to the agents affected by disturbance signals, i.e. $V_D = \{3, 5\}$. We are interested in decoupling the outputs $x_1(t) - x_2(t)$ and $x_4(t) - x_6(t)$ from the disturbance. Hence, the output variables in this case is given by $y = R^\top x$ where

$$R^\top = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

and $x \in \mathbb{R}^8$. Then π_R and π_S are given by:

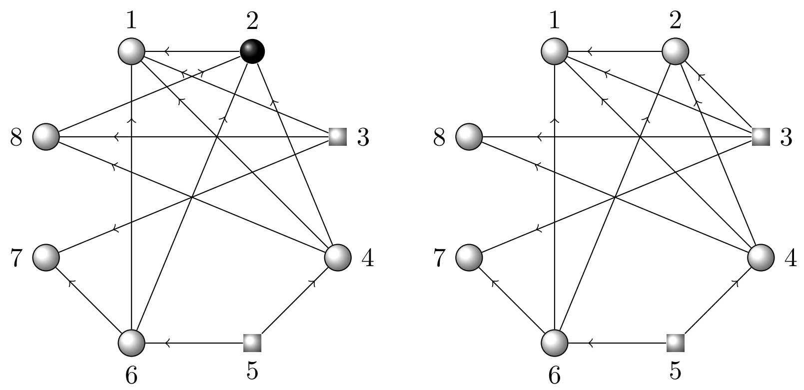

Figure 5.3: *The simple directed graph G (left) and H (right) of a diffusively coupled multi-agent network*

$$\pi_R = \{\{1,2\},\{3\},\{4,6\},\{5\},\{7\},\{8\}\}$$
$$\pi_S = \{\{1,2,4,6,7,8\},\{3\},\{5\}\}$$

By applying the algorithm given in Theorem 2.4.10, we obtain that

$$\pi^*_{\mathrm{AEP}}(\pi_s) = \{\{1\},\{2\},\{3\},\{4,6\},\{5\},\{7,8\}\}.$$

Then we easily conclude from Theorem 5.4.2 that the open-loop network (5.7) is not disturbance decoupled. However, given V_{L}, we can find an almost equitable partition with respect to V_{L} as follows

$$\pi = \{\{1,2\},\{3\},\{4,6\},\{5\},\{7,8\}\}. \tag{5.28}$$

Then we conclude from Theorem 5.4.5 that DDP of the network (5.2) is solvable by a state feedback.

To construct a state feedback, note that the partition π in (5.28) becomes almost equitable in a graph H as shown in Figure 5.3 (right). The graph H is obtained from graph G by removing the arc from vertex 8 to 2, and adding an arc from 3 to 2. As a result, by (5.25), the state feedback which solves the DDP of the network (5.2) is given by

$$u(t) = (x_3(t) - x_2(t)) - (x_8(t) - x_2(t)) = x_3(t) - x_8(t).$$

5.6 Concluding remarks

We have studied the disturbance decoupling problem (DDP) for diffusively coupled networks for both the open-loop and the closed-loop case. For both cases, we have developed graph theoretical sufficient conditions to solve DDP in terms of almost equitable partitions with respect to a cell. Such a class of graph partitions can be taken as generalizations of almost equitable partitions in Chapter 2 and as a graph theoretical translation of a class of controlled invariant subspaces of networks. In case the DDP is solvable, a solution state feedback has been synthesized, which is based on only the relative information of the states of the agents.

Chapter 6

Conclusions and further research

6.1 Conclusion

In this thesis, we have studied three system theoretical properties of diffusively coupled networks, namely controllability, partial consensus and disturbance decoupling. As a common thread of this thesis, we have provided graph theoretical characterizations for these system theoretical properties of diffusively coupled networks in terms of graph partitions and their characteristic matrices. The main contribution of this thesis can be summarized as follows:

- In Chapter 2, we have revealed the role that general linear dynamics of agents and the network topology play in controllability of diffusively coupled networks. Then we have provided both a lower and an upper bound for the controllable subspace of single-integrator networks in terms of *distance partitions* and *almost equitable partitions* respectively, by employing a combination of ideas from linear geometric control theory and graph theory. To the author's best knowledge, there was no general lower bound that appeared in the literature whereas an upper bound (see e.g. [31]) has been provided only for single-leader cases. Moreover, the geometric approach we take in deriving the upper bound in terms of almost equitable partitions provides a much stronger insight than those approaches in [31] or other work in which similar approaches have been adopted. Also, we have developed an algorithm that computes the maximal almost equitable partition involved in the upper bound. Further, we have shown that the bounds we have provided are tight, i.e., the bounds cannot be improved in terms of characteristic matrices of graph partitions. When the network topology has the so-called distance regularity property, we have provided a lower as well as an upper bound for the number of leaders one should choose for controllability of diffusively coupled networks.

- In Chapter 3, we have developed conditions for controllability of networks with switching topologies by extending the results in Chapter 2. We have clarified the role that linear dynamics of agents and the *switching* network topology play in controllability. Then we have constructed both the lower and the upper bounds for the controllable subspace and illustrated that the bounds are tight. Also, we have provided an algorithm to compute the upper bound.

- In Chapter 4, we have first derived necessary and sufficient algebraic conditions in terms of Laplacian matrices of both position and velocity graphs as well as output matrices for partial consensus of heterogeneous networks with double-integrator (position/velocity) agents. Then we have developed graph theoretical necessary conditions in terms of a class of almost equitable partitions.

- In Chapter 5, we have obtained graph theoretical conditions for disturbance decoupling problem (DDP) of both open-loop and closed-loop networks. To provide sufficient conditions for close-loop networks, we have introduced and developed a class of generalized almost equitable partitions called *almost equitable partitions with respect to a cell*, which provide graph theoretical characterizations of controlled invariant subspaces and also a strategy of synthesizing a solution for DDP of networks.

6.2 Further research topics

We have developed graph theoretical conditions for system theoretical properties of diffusively coupled networks. To do that, we mainly focus on those subspaces that are induced by the images of characteristic matrices of partitions. Unfortunately, not all the subspaces can be described in terms of partitions. As such, one may not find a partition that induces exactly the controllable subspace in general. As a result, although the controllability of a network is fully characterized by the controllable subspace, it cannot be *fully* characterized by any partition of the graph in general cases. An interesting open question is whether one can generalize the concept of partition (for instance by allowing a vertex to belong to more than one different cell) in order to capture all possible subspaces that can appear as controllable subspaces for given graphs and leader sets.

As we have developed a systematic way of choosing leaders for networks with distance regular topologies, we are further interested in determining a way that leads to the *minimal* number of leaders. Moreover, one can investigate systematic ways of choosing leaders for networks with general topologies.

The partial consensus problem we have studied is for networks with double-integrator agents. It is interesting to consider this problem for networks with general second-order agents. For instance, each agent has mass-damper-spring dynamics.

Moreover, the conditions we have developed are based on an assumption that all underlying graphs are undirected. One can try to extend the results for undirected graphs to directed ones.

When solving DDP of networks, we have not required the closed-loop network to be stable. Investigation of DDP with stability is yet another open problem for future research.

Bibliography

[1] A. Arenas, A. Diaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou. Synchronization in complex networks. *Physics Reports*, 469(3):93–153, 2008.

[2] G. Basile and G. Marro. *Controlled and Conditioned Invariants in Linear System Theory*. College of Engineering, University of California, 1968.

[3] G. Basile and G. Marro. Controlled and conditioned invariant subspaces in linear system theory. *Journal of Optimization Theory and Applications*, 3(5):306–315, 1969.

[4] D. S. Bernstein. *Matrix Mathematics: Theory, Facts and Formulas with Application to Linear Systems*. Princeton University Press, Princeton and Oxford, 2003.

[5] T. Borsche and S. A. Attia. On leader election in multi-agent control systems. In *Proc. of the Chinese Control on Decision Conference*, pages 102–107, 2010.

[6] M. E. Broucke. Disjoint path algorithms for planar reconfiguration of identical vehicles. In *Proc. of the American Control Conference*, pages 2199–2204, 2003.

[7] M. E. Broucke. Reconfiguration of identical vehicles in 3d. In *Proc. of the IEEE Conference on Decision and Control*, pages 1538–1543, 2003.

[8] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance Regular Graphs*. Springer-Verlag, Berlin, Heidelberg, 1989.

[9] F. Bullo, J. Cortes, and S. Martinez. *Distributed Control of Robotic Networks*. Princeton University Press, Princeton, 2009.

[10] D. Burgarth, D. D'Alessandro, L. Hogben, S. Severini, and M. Young. Zero forcing, linear and quantum controllability for systems evolving on networks. *IEEE Transactions on Automatic Control.* To appear.

[11] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra*. Springer-Verlag, New York, 1981.

[12] M. K. Camlibel and S. Zhang. Partial consensus for heterogenous multi-agent systems with double integrator dynamics. In *Proc. of the IFAC Conference on Analysis and Design of Hybrid Systems*, pages 121–126, 2012.

[13] M. K. Camlibel, S. Zhang, and M. Cao. Comments on 'Controllability analysis of multi-agent systems using relaxed equitable partitions'. *International Journal of Systems, Control and Communications*, 4(1/2):72–75, 2012.

[14] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment: A graphical approach. *SIAM Journal on Control and Optimization*, 47(2):575–600, 2008.

[15] M. Cao, S. Zhang, and M. K. Camlibel. A class of uncontrollable diffusively coupled multi-agent systems with multi-chain topologies. *IEEE Transactions on Automatic Control*, 58(2):465–469, 2013.

[16] D. M. Cardoso, C. Delorme, and P. Rama. Laplacian eigenvectors and eigenvalues and almost equitable partitions. *European Journal of Combinatorics*, 28(3):665–673, 2007.

[17] A. Chapman and M. Mesbahi. On strong structural controllability of networked systems: a constrained matching approach. In *Proc. of the American Control Conference*, pages 6126–6131, 2013.

[18] W. B. Dunbar and R. M. Murray. Receding horizon control of multi-vehicle formations: a distributed implementation. In *Proc. of the IEEE Conference on Decision and Control*, pages 1995–2002, 2004.

[19] M. Egerstedt, S. Martini, M. Cao, M. K. Camlibel, and A. Bicchi. Interacting with networks: How does structure relate to controllability in single-leader, consensus network? *Control Systems Magazine*, 32(4):66–73, 2012.

[20] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer-Verlag, New York, 2001.

[21] D. Goldin, S. A. Attia, and J. Raisch. Consensus for double integrator dynamics in heterogeneous networks. In *Proc. of the IEEE Conference on Decision and Control*, pages 4504–4510, 2010.

[22] D. Goldin and J. Raisch. Controllability of second order leader-follower systems. In *Proc. of the IFAC Workshop on Distributed Estimation and Control in Networked Systems*, pages 233–238, 2010.

[23] T. Takamatsu I. Hashimoto and Y. Nakai. Decoupling and pole assignment in linear multivariable systems: a geometric approach. *Automatica*, 15(4):387–402, 1979.

[24] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

[25] F. Jiang, L. Wang, G. Xie, Z. Ji, and Y. Jia. On the controllability of multiple dynamic agents with fixed topology. In *Proc. of the American Control Conference*, pages 5665–5670, 2009.

[26] V. Kumar, N. E. Leonard, and A. S. Morse. *Cooperative Control*. Springer-Verlag, New York, 2005.

[27] Z. K. Li, Z. S. Duan, G. R. Chen, and L. Huang. Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint. *IEEE Transactions on Circuits and Systems*, 57(1):213–224, 2010.

[28] B. Liu, T. Chu, L. Wang, and G. Xie. Controllability of a leader-follower dynamic network with switching topology. *IEEE Transactions on Automatic Control*, 53(4):1009–1013, 2008.

[29] Y. Y. Liu, J. J. Slotine, and A. L. Barabasi. Controllability of complex networks. *Nature*, 473(7346):161–173, 2011.

[30] W. Lu, B. Liu, and T. Chen. Cluster synchronization in networks of coupled nonidentical dynamical systems. *Chaos*, 20(1):013120, 2010.

[31] S. Martini, M. Egerstedt, and A. Bicchi. Controllability analysis of multi-agent systems using relaxed equitable partitions. *International Journal of Systems, Control and Communications*, 2(1/2/3):100–121, 2010.

[32] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton and Oxford, 2010.

[33] N. Monshizadeh, H. L. Trentelman, and M. K. Camlibel. Synchronization analysis of multi-agent systems with switching topologies. In *Proc. of the International Symposium on Mathematical Theory of Networks and Systems*, page 0169, 2012.

[34] N. Monshizadeh, S. Zhang, and M. K. Camlibel. Disturbance decoupling problem for multi-agent systems: a graph topological approach. *IEEE Transactions on Control of Network Systems*. Submitted.

[35] N. Monshizadeh, S. Zhang, and M.K. Camlibel. Zero forcing sets and controllability of systems defined on graphs. *IEEE Transactions on Automatic Control*. Accepted.

[36] M. Nabi-Abdolyousefi and M. Mesbahi. On the controllability properties of circulant networks. *IEEE Transactions on Automatic Control*. To appear.

[37] R. Olfati-Saber. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.

[38] R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.

[39] R. Olfati-Saber and R. M. Murray. Agreement problems in networks with directed graphs and switching topology. In *Proc. of the IEEE Conference on Decision and Control*, pages 4126 – 4132, 2003.

[40] G. Parlangeli and G. Notarstefano. On the reachability and observability of path and cycle graphs. *IEEE Transactions on Automatic Control*, 57(3):743–748, 2012.

[41] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt. Controllability of multi-agent systems from a graph theoretic perspective. *SIAM Journal on Control and Optimization*, 48(1):162–186, 2009.

[42] W. Ren. Second-order consensus algorithm with extensions to switching topologies and reference models. In *Proc. of the American Control Conference*, pages 1431–1436, 2007.

[43] W. Ren. On consensus algorithms for double-integrator dynamics. *IEEE Transactions on Automatic Control*, 58(6):1503–1509, 2008.

[44] W. Ren and E. Atkins. Second-order consensus protocols in multiple vehicle systems with local interactions. In *AIAA Guidance, Navigation and Control Conference and Exhibit*, 2005.

[45] W. Ren, R. W. Beard, and E. M. Atkins. Information consensus in multivehicle cooperative control. *IEEE Control System Magazine*, 27(2):71–82, 2007.

[46] W. Ren and W. R. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005.

[47] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 14(11):2557–2562, 2009.

[48] F. Sorrentino, M. di Bernardo, F. Garofalo, and G. Chen. Controllability of complex networks via pinning. *Physical Review E*, 75(4):046103, 2007.

[49] Z. Sun and S. S. Ge. *Switched Linear Systems: Control and Design*. Springer, London, 2005.

[50] Z. Sun, S. S. Ge, and T. H. Lee. Controllability and reachability criteria for switched linear systems. *Automatica*, 38(5):775–786, 2002.

[51] H. G. Tanner. On the controllability of nearest neighbor interconnections. In *Proc. of the IEEE conference on Decision and Control*, pages 2467–2472, 2004.

[52] H. L. Trentelman, A. Stoorvogel, and M. Hautus. *Control Theory of Linear Systems*. Springer-Verlag, London, 2001.

[53] H. L. Trentelman, K. Takaba, and N. Monshizadeh. Robust synchronization of uncertain linear multi-agent systems. *IEEE Transactions on Automatic Control*, 58(6):1511–1523, 2013.

[54] T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen, and O. Shochet. Novel type of phase transition in a system of self-driven particles. *Physical Review Letters*, 75(6):1226–1229, 1995.

[55] W. M. Wonham. *Linear Multivariable Control: A Geometric Approach*. Springer-Verlag, New York, 1985.

[56] W. M. Wonham and A. S. Morse. Decoupling and pole assignment in linear multivariable systems: a geometric approach. *SIAM Journal of Control*, 8(1):1–18, 1970.

[57] W. Wu, W. Zhou, and T. Chen. Cluster synchronization of linearly coupled complex networks under pinning control. *IEEE Transactions on Circuits and Systems I*, 56:829–839, 2009.

[58] W. Xia and M. Cao. Clustering in diffusively coupled networks. *Automatica*, 47(11):2395–2405, 2011.

[59] A. Y. Yazicioglu, W. Abbas, and M. Egerstedt. A tight lower bound on the controllability of networks with multiple leaders. In *Proc. of the IEEE conference on Decision and Control*, pages 1978–1983, 2012.

[60] W. Yu, G. Chen, and M. Cao. On second-order consensus in multi-agent dynamical systems with directed topologies and time delays. In *Proc. of the IEEE Conference on Decision and Control*, pages 3709–3714, 2009.

[61] W. Yu, G. Chen, and M. Cao. Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics. *IEEE Transactions on Automatic Control*, 40(3):881–891, 2010.

[62] W. Yu, G. Chen, and M. Cao. Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems. *Automatica*, 46(6):1089–1095, 2010b.

[63] S. Zhang, M. K. Camlibel, and M. Cao. Controllability of diffusively-coupled multi-agent systems with general and distance regular topologies. In *Proc. of the IEEE conference on Decision and Control and 2011 European Control Conference*, pages 759–764, 2011.

[64] S. Zhang, M. Cao, and M. K. Camlibel. Upper and lower bounds for controllable subspaces of networks of diffusively coupled agents. *IEEE Transactions on Automatic Control*. To appear.

[65] S. Zhang, M. Cao, and M. K. Camlibel. On the controllability of diffusively coupled multi-agent networks with switching topologies. In *Proc. of the International Symposium on Mathematical Theory of Networks and Systems*, page 0167, 2012.

Summary

This thesis addresses three problems for distributed control of diffusively coupled networks: controllability, partial consensus and disturbance decoupling. In particular, graph theoretical characterizations for the solvability of these problems are provided through partitioning network agents in certain ways. The three problems studied and the graph theoretical characterizations can be summarized as follows.

Controllability of a diffusively coupled network indicates whether all the agents (or the overall network) can be made to behave in some desirable way by suitably manipulating only some of the agents (leaders). The role that general linear dynamics of agents and the network topology play in controllability of diffusively coupled networks is clarified. If the dynamics of each agent is single-integrator, then controllability of a diffusively coupled network can be characterized in terms of the so-called distance partitions and almost equitable partitions. By taking a geometric approach, the characterization in terms of almost equitable partitions become much easier to conclude and to compute for more complex cases than in existing results. The study in this thesis also shows that there is no single partition of which the characteristic matrix can completely reveal controllability of diffusively coupled networks except that the network topology is known as a priori. Moreover, when the network topology is described by distance regular graphs, a strategy of choosing leaders is provided so that the network becomes controllable with the chosen leaders. Most of the above results can be extended without much difficulty to controllability of diffusively coupled networks when the network topology is switching arbitrarily within a finite set of admissible topologies.

The consensus problem of diffusively coupled networks has been well-known in the literature. One chapter of this thesis addresses a different version for the consensus problem: partial consensus of heterogeneous networks with double-integrator (position/velocity) agents. By "heterogeneous", we mean the topologies arising from the communication between positions and velocities of agents are not necessarily the same. Here, "Partial consensus" indicates that a pre-specified set of positions and velocities reach an agreement in contrast to the consensus problem for which all positions and velocities are required to reach an agreement. Necessary and sufficient algebraic conditions are concluded. Moreover, by employing the geometric control theory, we also conclude graph theoretical necessary conditions in terms of a class of

almost equitable partitions.

In the presence of disturbance signals, it is natural to investigate whether a certain output of the network is not affected by the disturbance. This motivates the study of disturbance decoupling problem (DDP) of diffusively coupled networks. The graph theoretical conditions to solve DDP of diffusively coupled networks are provided for both open- and closed-loop cases. With the aid of geometric control theory, a graph theoretical sufficient condition to solve DDP is presented in terms of a class of generalized almost equitable partitions called *almost equitable partitions with respect to a cell*. This class of generalized almost equitable partitions can be considered as graph theoretical characterizations of the well-known controlled invariant subspaces of linear control theory and provide a strategy to synthesize a solution (if possible) for DDP of networks.

Samenvatting

Dit proefschrift richt zich op drie problemen voor de gedistribueerde aansturing van diffuus gekoppelde netwerken: regelbaarheid, partiële consensus en storingsontkoppeling. Er worden grafentheoretische karakteriseringen verstrekt voor de oplosbaarheid van deze problemen door de agenten in het netwerk op een bepaalde manier te verdelen. De drie bestudeerde problemen en de grafentheoretische karakteriseringen kunnen als volgt worden samengevat.

Regelbaarheid van een diffuus gekoppeld netwerk geeft aan of het mogelijk is om het gedrag van alle agenten (of het gehele netwerk) aan te sturen door slechts een aantal agenten (leiders) te beïnvloeden. Daarbij is de rol die de algemene lineaire dynamica van de agenten speelt in de regelbaarheid van diffuus gekoppelde netwerken verduidelijkt. Als de dynamica van elke agent een enkele integrator is, dan kan de regelbaarheid van een diffuus gekoppeld netwerk worden gekarakteriseerd in termen van de zogenoemde afstand partities en bijna billijke partities. Door een geometrische aanpak is de karakterisering in termen van bijna billijke partities veel eenvoudiger voor complexere gevallen te bepalen en te berekenen dan in bestaande resultaten. De studie in dit proefschrift laat ook zien dat er geen enkele partitie bestaat waarvoor de karakteristieke matrix de regelbaarheid van de diffuus gekoppelde netwerken volledig kan openbaren, behalve wanneer de netwerktopologie van tevoren bekend is. Bovendien wordt voor afstandsreguliere grafen een strategie voor het kiezen van leiders gegeven, zodanig dat het netwerk regelbaar wordt met de gekozen leiders. Het merendeel van bovenstaande resultaten kan moeiteloos worden gegeneraliseerd naar de regelbaarheid van diffuus gekoppelde netwerken, waarvan de netwerk topologie arbitrair schakelt binnen een eindige verzameling van toegestane topologiën.

Het consensusprobleem van diffuus gekoppelde netwerken is welbekend in de literatuur. Één hoofdstuk in dit proefschrift richt zich op een andere versie van het consensus probleem: partiële consensus van heterogene netwerken met dubbele integrator (positie/snelheid) agenten. Met "heterogeen" bedoelen we dat de topologiën die voortkomen uit de communicatie aangaande posities en snelheden van de agenten niet persé hetzelfde zijn. Met "partiële consensus" geven we aan dat een vooraf gespecificeerde verzameling van posities en snelheden overeenstemming bereiken, in tegenstelling tot het consensus probleem waarvoor alle posities en snelheden overeenstemming dienen te bereiken. Noodzakelijke en voldoende algebraïsche

voorwaarden zijn hiervoor opgesteld. Bovendien stellen we ook noodzakelijke grafentheoretische condities op in termen van een klasse van bijna billijke partities, door gebruik te maken van geometrische regeltechniek.

Wanneer storingssignalen aanwezig zijn, is het natuurlijk om te onderzoeken of een bepaalde uitgang van het netwerk niet beïnvloed wordt door de verstoring. Dit is de aanleiding voor de studie van het storingsontkoppeling probleem (SOP) voor diffuus gekoppelde netwerken. De grafentheoretische voorwaarden om het SOP voor diffuus gekoppelde netwerken op te lossen zijn gegeven voor zowel het open als het gesloten lus systeem. Met behulp van geometrische regeltechniek is een voldoende grafentheoretische voorwaarde opgesteld om het SOP op te lossen in termen van een klasse van gegeneraliseerde bijna billijke partities, de zogenaamde *bijna billijke partities ten opzichte van een cel*. Deze klasse van gegeneraliseerde bijna billijke partities kan worden beschouwd als een graaf theoretische karakterisering van de welbekende regelbare invariante deelverzamelingen uit de lineaire regeltechniek. Ze verschaffen een strategie om (indien mogelijk) een oplossing op te stellen voor SOP van netwerken.