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Abstract The paper works with a formal model of referenda, where a finite number of voters can choose between two options and abstention. A referendum will be invalid if too many voters abstain, otherwise the referendum will return one of the two options. We consider quorum rules where an option is chosen if it is preferred by the majority of voters and if at least a certain number of voters (the quorum) votes for the alternative. The paper characterizes these rules as the only referenda which are strategy-proof over certain preferences.

Keywords Social choice theory · Referendum · Voting rules · Strategy-proofness

JEL classification D71 · D72

1 Introduction

Referenda are used in a number of countries on local, regional, and national levels. To give some examples from European countries, Italy knows referenda on a national level, whereas Germany only allows for referenda on the state level. In the Netherlands, in spite of the recent 2005 referendum on the European constitution, only referenda on the local city level are institutionalized.

Besides this difference in political scope, referenda differ also on a number of other dimensions. Referenda can be initiated by either the citizens or by the elected representatives. They can be corrective in trying to correct a law that has been passed, or they can propose a new law. They can be binding or non-binding, in which case

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elected representatives are not bound to act or vote in accordance with the referendum. Finally, it matters how many options are available to choose from in a referendum. The most common case is the situation where a law is proposed (or opposed to) and eligible voters have the choice between voting yes, voting no, and abstaining. This is also the situation we will consider in this paper.

A final difference concerns the quorum requirement that is imposed in many referenda. There are countries such as Estonia and Switzerland where a referendum is valid no matter how many people cast their vote in the referendum. However, in many cases a referendum is valid only if a certain participation quorum has been met. In Italy, 50 % of the eligible voters need to cast their vote in a referendum for the referendum to be valid, similarly in Poland and Portugal (Aguiar-Conraria and Magalhães 2010b). For local referenda in the Netherlands, the quorum depends on the city but is usually lower than 50 %, e.g., 30 % in the city of Rotterdam. Situations also differ in what happens when a referendum is invalid. In many cases, one of the two options is a default option that represents the status quo and that will be adopted (maintained) if the referendum turnout is too low. In other cases, the consequences of an invalid referendum are uncertain. In the case of Dutch local referenda, the city council will then vote on the issue with an (at least formally) uncertain outcome. This issue is discussed in more detail in Sect. 3 because it yields two different models of referenda, a symmetric model with three outcomes (treated in Sect. 5) and an asymmetric model with only two outcomes (treated in Sect. 6).

A problem with the participation quorum is the so-called *no-show paradox* (Fishburn and Brams 1983): a voter who is against a proposed change in the status quo and who would thus naturally vote no, might be faced with a situation where a vocal minority is in favor of changing the status quo. If he also thinks that turnout for the referendum will be low because the issue at stake is not sufficiently important for most people to participate in the referendum, he might decide to abstain rather than to vote no because that way he thinks he is more likely to maintain the status quo, obtaining an invalid referendum rather than a valid referendum yielding no.

This form of strategic manipulation evidenced by the no-show paradox is undesirable for at least four reasons. First, for an institutional designer, the information revealed by the referendum becomes highly ambiguous because we do not know how to interpret abstentions anymore. Second, this kind of manipulation increases the cognitive burdens for the agents involved because they will need to obtain information about how many people are likely to vote for and against, and in the end these strategic calculations may turn out to be wrong. Third, we would like a referendum procedure that induces people to honestly reveal their top preferences instead of voting strategically. Fourth, we would like a referendum procedure to encourage participation rather than abstention, also because in political practice, low participation rates are usually considered problematic for the legitimacy of the decisions reached. Hence, we would like a referendum procedure that is non-manipulable, a procedure where the no-show paradox cannot arise.

As it turns out, there are referendum procedures which do not suffer from the no-show paradox. These procedures also work with a quorum but not with a participation quorum. Rather, in the case where we are dealing with the choice between change and the status quo, these procedures require that for change to be chosen, a certain minimal number of people needs to vote for change. In such an *acceptance quorum* procedure, the quorum requirement is thus not imposed on the number of total voters, but only on the number of voters who vote for change. Under this alternative quorum rule the voter we just considered cannot obtain a better result from strategically abstaining, for a no vote is not counted towards meeting the quorum.

A number of countries work with acceptance quorums rather than participation quorums, such as Denmark and some German states. But the fact that a referendum working with an acceptance quorum is somewhat non-manipulable (in a sense to be made precise) and in that sense preferable to a participation quorum raises another question: are there possibly other referendum procedures that are non-manipulable as well? Which is the class of referendum procedures that is strategy-proof in situations like the one described? In what follows we will develop a formal model that allows us to answer this question.

After reviewing previous formal work on referenda in Sect. 2, I will discuss how many policy outcomes we should use in our model of referenda (Sect. 3). The formal model of referenda and the main axioms are presented in Sect. 4, together with some preliminary results. The situation where an invalid referendum yields a genuine third alternative not equivalent to the status quo will be considered in Sect. 5. This model will be used to prove the first of our two main results, Theorem 1, which characterizes majority quorum rules in terms of strategy-proofness. Section 6 then deals with the situation where an invalid referendum means the adoption of the status quo and proves the second main result, Theorem 2, which provides a characterization of acceptance quorum rules. Proofs can be found in the appendix.

2 Previous work on modeling referenda

There have been other attempts to formally model referenda. These models have been developed mainly within game theory and social choice theory. This literature will be surveyed briefly and related to the approach taken in this paper.

Starting with game-theoretic models, Herrera and Mattozzi (2010) develop an equilibrium model which exhibits the paradoxical result that in equilibrium, the participation level reaches the quorum only if there is no quorum imposed. Zwart (2010) investigates the theoretical and practical difficulties in setting the quorum level so that representativeness of the referendum result is guaranteed. Game-theoretic models are also used by Aguiar-Conraria and Magalhães (2010a) and Maniquet and Morelli (2010) to investigate the effects of different quorum rules.

There has also been interesting empirical work comparing the effects of different quorum rules. Aguiar-Conraria and Magalhães (2010b) statistically compare the effect of different quorum rules looking at different European countries. In the terminology of this paper, their results provide an empirical argument for choosing an acceptance quorum rather than a participation quorum, since (only) the latter has a negative effect on turnout.

Turning from game theory to social choice theory, one can view referenda simply as voting procedures and try to apply the well-known voting paradoxes and impossibility results to referenda, as done by Nurmi (1997, 1998). Laruelle and Valenciano (2011) use a quaternary model where voters can vote yes, no, stay home or cast a blank ballot. Various quorum rules are considered in this framework. The model to be considered in this paper does not distinguish staying home from casting a blank ballot, as will be explained further on. Côrte-Real and Pereira (2004) focus on the no-show paradox and investigate different quorum rules, including what in this paper is called the acceptance quorum. Furthermore, they address the question whether one can design a referendum whose result is representative of the population as a whole, and they demonstrate an impossibility result.

The participation quorum has been characterized axiomatically by Houy (2009), and Houy (2007) also characterized participation quorum rules that require also the difference between yes-votes and no-votes to be above a certain threshold. His work can be seen as following in the tradition of May's famous characterization of majority voting (May 1952). My paper lies in the same tradition, providing characterization results for acceptance quorum rules (Theorem 2) as well as majority quorum rules (Theorem 1). To the best of my knowledge, these are the first characterization results for these two quorum rules. Furthermore, in contrast to other characterization results, we focus on the question of manipulability or strategy-proofness, and our aim is a set of axioms with normative appeal for choosing between different quorum rules.

Houy (2009) characterizes participation quorum rules using the axioms of anonymity, neutrality, weak Pareto optimality, two monotonicity conditions and a minimal decisiveness axiom. However, the characterization result provides no argument for the adoption of this type of quorum rules in actual referendum practice, since the normative appeal of particularly the minimal decisiveness axiom is weak. In fact, Houy makes no claim in his paper that his result should be used as an argument for using these quorum rules.

Comparing my results to May's result, the class of majority quorum rules includes May's majority rule, and hence the combination of his axioms is logically stronger than the axiomatization presented in this paper. May's axioms are universal domain, anonymity, neutrality and positive responsiveness. In order to characterize the larger class of majority quorum rules, our axiomatization replaces positive responsiveness with three axioms: weak Pareto optimality, a quorum axiom and a minimal strategy-proofness axiom. Given that weak Pareto optimality and the quorum axiom are very plausible axioms under the referendum interpretation we are considering, our characterization result can be viewed as a characterization of the referenda which satisfy a minimal non-manipulation requirement. As a consequence, this work can serve to justify adopting majority quorum rules (or acceptance quorum rules) in the practice of referendum legislature.

3 How many outcomes does a referendum have?

A crucial modeling choice when dealing with referenda concerns the set of possible outcomes of a referendum. First of all, there have been referenda where voters could choose between more than two options. As an example, in 2007 the Dutch city of Arnhem had a referendum where citizens could choose between three alternatives, three different models for the construction of a port. In this paper, we will not be concerned with these kinds of referenda; we will assume that voters get to choose between two alternatives on a referendum ballot.

But even in the case of a choice between two alternatives, change and the status quo, we still need to choose how to model the possible referendum outcomes. Assuming anonymity, we can describe the result of a referendum by giving the number of people who voted for change (c), the number of people who voted for the status quo (q), the number of people who submitted an invalid (or empty) ballot (x), and the number of people who did not vote (y). Hence, a referendum result would be a four-tuple (c, q, x, y). The referendum literature abstracts from these detailed results by mapping these four-tuples into a small set of policy outcomes. Here, for the purposes for this paper, we can distinguish two situations.

First, we can consider a model with just two policy outcomes which correspond to the alternative actually implemented based on the referendum. Usually, this will mean that change occurs if and only if the quorum is met and the majority votes for change. Otherwise, the status quo remains in place. This model is explored in Sect. 6. Second, we can consider a more general model with three policy outcomes: if the quorum is met, the majority determines whether change occurs (outcome 1) or the status quo is retained (outcome -1). If the quorum is not met, however, a different third policy outcome obtains (outcome 0). This is the model explored in Sect. 5. How can we think about this third policy outcome which is different from the status quo? Houy (2009, p. 296) states that "if the quorum is not reached or if no majority exists, the society abstains. Our interpretation for abstention of the society is the following: when the society abstains, the choice will depend on anything else but the vote (agenda for instance)". I will comment on this third policy outcome from a conceptual and from a practical point of view.

Conceptually, first, the referendum outcome where the status quo is maintained because the quorum was not reached may have different political consequences from a referendum where the quorum was reached but the status quo received a majority. When the quorum is reached, the matter will often be considered as settled, but when the quorum is not reached, especially if it is almost reached and a large majority is for change, further political actions might follow. Second, voters may not consider the outcomes -1 and 0 to be the same. A voter who favors direct democracy may strictly prefer all outcomes where the quorum is reached (1 and -1) to an outcome where the quorum is not reached (0). Hence, looking at the possible voter preferences, it makes sense to distinguish these three outcomes.

Practically, as was already mentioned in the introduction, local referenda in the Netherlands exemplify this situation of three possible policy outcomes. When the city government proposes a policy measure such as a new construction project, citizens can call for a referendum. When a referendum is held, the city council postpones voting on the policy measure until after the referendum has been held. In the referendum, citizens can vote for or against the policy measure, but the referendum is considered to be valid only if a specific quorum has been met.[1] If the quorum is not met and the referendum is hence invalid, the city council will vote on the policy mea-

[1] While local referenda in the Netherlands are always non-binding to the city council, the city council will usually self-bind itself to the referendum outcome provided the quorum is met.

sure, and the result of this vote may or may not reflect the majority opinion of the referendum. Hence, the political and policy consequences of the three referendum outcomes differ.

4 Formal model and axioms

4.1 Formal model and interpretation

The formal model we will use here is essentially that of Houy (2009). For $n \geq 1$, let $\{1, \ldots, n\}$ be the set of possible voters, e.g., the individuals legally eligible to vote in a society. Each individual can choose between abstaining, voting yes and voting no. We will represent these possible choices as the set of possible votes or ballots $\{\text{-}1, 0, 1\}$ where 0 represents abstention, 1 represents voting yes and -1 represents voting no. A vote vector $V = (v_1, \ldots, v_n)$ contains the vote choices of all individuals, so $V \in \{\text{-}1, 0, 1\}^n$ and for all $i \leq n$ we have $v_i \in \{\text{-}1, 0, 1\}$. We denote the set of all vote vectors as $\mathcal{V}^n = \{\text{-}1, 0, 1\}^n$, or simply as $\mathcal{V}$ if n is arbitrary.

Given a vote vector $V = (v_1, \ldots, v_n)$, we let $n^+(V) = |\{i \leq n | v_i = 1\}|$ denote the number of individuals in V voting yes. Similarly, we have $n^-(V) = |\{i \leq n | v_i = \text{-}1\}|$ and $n^0(V) = |\{i \leq n | v_i = 0\}|$. Furthermore, if σ is a permutation of $\{1, \ldots, n\}$, we define $V_\sigma = (v_{\sigma(1)}, \ldots, v_{\sigma(n)})$. Finally, we write $V[i{:}k] = (v_1, \ldots, v_{i-1}, k, v_{i+1}, \ldots, v_n)$ for the vote vector just like V except that individual i votes k.

A referendum rule $C^n : \mathcal{V}^n \rightarrow \{\text{-}1, 0, 1\}$ maps vote vectors V into a referendum outcome. In this case, 0 will refer to an invalid referendum, whereas 1 refers to a valid referendum where yes wins, and -1 refers to a valid referendum where no wins. Again, we usually will write C instead of C^n. Note that the numbers -1, 0 and 1 are used in two different ways: They can represent either a vote (a ballot) when used as an input to a referendum rule, or a referendum outcome when used as the output of a referendum rule. The context will always clarify the intended meaning.

The simplicity of the model just defined is achieved based on a simplifying assumption. We do not distinguish staying at home from submitting a blank ballot. Both of these actions are modeled by 0 in the input profile which we will always interpret as abstention. This simplification can be justified on three grounds. (1) It simplifies the model, (2) the number of people who submit blank ballots is usually very small, especially when compared to the number of people staying at home, and (3) for the quorum rules characterized in this paper, there is indeed no difference between staying at home and submitting a blank ballot.

Note that a referendum rule is essentially a game form, where the votes or ballots submitted are the players' strategies. In the following section, we will discuss the notion of manipulability or strategy-proofness for referendum rules. However, as observed already by Gibbard (1973), for game forms alone there is no such thing as manipulation, since strategies are in general not preference relations. In particular, in referendum rules, strategies are ballots and not preference relations over outcomes. Hence, as done by Gibbard, we will need to associate referendum strategies to preference relations before being able to talk about manipulation.

Consider again the set of referendum outcomes {-1, 0, 1}. Two preference relations $\geq_1, \geq_{-1} \subseteq$ {-1, 0, 1} × {-1, 0, 1} over these outcomes will be of crucial importance later on. $\geq_1$ is the preference relation of the voter who favors yes winning, over an invalid referendum over no winning, i.e., we have $1 >_1 0 >_1 -1$. The relation $\geq_{-1}$ is the converse relation, i.e., $-1 >_{-1} 0 >_{-1} 1$. Together with the voter who is indifferent between the three possible referendum outcomes, these two preference relations over outcomes arguably represent the two most common types of voters in referenda. Of course, other types of voters are possible: as was already mentioned, we can imagine a voter who prefers any valid referendum over an invalid one, because she cares about participatory democracy most of all. We will come back to this kind of voter in the concluding section.

4.2 Axioms and preliminary results

We now consider a number of axioms that may be imposed on referendum rules, also presenting some preliminary results useful for our main theorems to be presented later. The first three axioms of anonymity, neutrality and weak Pareto-optimality are standard and are also adopted by Houy (2009) (in contrast to the other axioms).

Axiom 1 *(A, Anonymity) A referendum rule C satisfies anonymity iff for all permutations σ of N, and all vote vectors $V \in \mathcal{V}$, $C(V) = C(V_\sigma)$.*

Axiom 2 *(N, Neutrality) A referendum rule C satisfies neutrality iff for all vote vectors $(v_1, \ldots, v_n) \in \mathcal{V}$, $C(-v_1, \ldots, -v_n) = -C(v_1, \ldots, v_n)$.*

Axiom 3 *(WP, Weak Pareto-optimality) A referendum rule C satisfies weak Pareto optimality iff $C(1, \ldots, 1) = 1$.*

The following three axioms will play a crucial role in the later characterization results. The first axiom requires that valid referenda yielding yes should reflect the majority decision of the voters.

Axiom 4 *(M, Majority) A referendum rule C satisfies the majority condition iff for all $V \in \mathcal{V}$, if $C(V) = 1$ then $n^+(V) > n^-(V)$.*

The following quorum axiom states that whenever a referendum is invalid in spite of there being a majority for one of the alternatives, then the referendum should remain invalid if more voters abstain. It thus captures the meaning of a quorum requirement.

Axiom 5 *(Q, Quorum) A referendum rule C satisfies the quorum condition iff for all $V \in \mathcal{V}$, if $C(V) = 0$ and $n^+(V) \neq n^-(V)$ then for all $V' \in \mathcal{V}$ such that $n^+(V') \leq n^+(V)$ and $n^-(V') \leq n^-(V)$ we have $C(V') = 0$.*

The claim is that this axiom captures our intuitive meaning of a quorum, not that it is the weakest possible axiom yielding the subsequent characterization result, nor that there are no plausible rules violating the axiom. For an example of a rule violating the axiom, consider the minimum-difference rule which returns 0 if the difference between yes- and no-votes is below a certain threshold and the majority opinion otherwise. This minimum-difference rule ties validity to a certain margin of victory among the votes that have been cast. The rule has been characterized by Llamazares (2006). Houy (2007) also characterizes this rule and furthermore also considers minimum-difference rules with added quorum requirements. These rules will also not satisfy our quorum axiom since the axiom states that a referendum can be invalid only because there is a tie or because the quorum has not been reached, not for any other reason.

Finally, we come to manipulability and strategy-proofness. Here, we are interested in two different requirements. The first axiom SP expresses that at least our two special types of voters must have a dominant strategy in the referendum game: they have ballot choices available to them which will be optimal no matter how the other voters vote.

Axiom 6 *(SP, minimal strategy-proofness) A referendum rule C is minimally strategy-proof iff for all $i \leq n$ and $R_i \in \{\geq_{-1}, \geq_1\}$ there is some ballot $v_* \in \{-1, 0, 1\}$ such that for all $V \in \mathcal{V}$, $C(V[i : v_*])\ R_i\ C(V)$.*

Minimal strategy-proofness does not say anything about what the dominant strategies are for our two special voter types. Ideally, we would hope that the dominant voting strategies are the sincere voting actions, i.e., for a voter with preference $\geq_1$ voting 1 is a dominant strategy, and for a voter with preference $\geq_{-1}$ voting -1 is a dominant strategy. This stronger requirement is formalized in the following axiom.

Axiom 7 *(SP*, minimal sincere strategy-proofness) A referendum rule C satisfies minimal sincere strategy-proofness iff for all $i \leq n$ and for all $V \in \mathcal{V}$ we have*

1. $C(V[i : 1]) \geq_1 C(V)$, *and*
2. $C(V[i : -1]) \geq_{-1} C(V)$.

Note that, the first condition of this axiom can be rewritten as is $C(V[i : 1]) \geq C(V)$ and the second condition as $C(V[i : -1]) \leq C(V)$. Furthermore, note that a referendum rule which satisfies minimal sincere strategy-proofness also satisfies minimal strategy-proofness, i.e., axiom SP* implies axiom SP. We will see later that the converse implication does not hold.

The consequences of strategy-proofness that are needed for the following characterization theorems are summarized in the following three lemmas. The first lemma states that minimal strategy-proofness together with anonymity, neutrality and weak Pareto-optimality implies that sincere voting is a dominant strategy for our two special types of voters.

Lemma 1 *(Sincere voting) Axioms A, N, WP, and SP together imply SP*.*

The following monotonicity lemma will be used a lot in our proofs and follows easily from Lemma 1.

Lemma 2 *(Monotonicity) If C is a referendum rule satisfying axioms A and SP*, then for any $V \in \mathcal{V}$ we can write $C(V)$ as $C(Y, N)$ with $Y = n^+(V)$ and $N = n^-(V)$ such that*

1. *if* $C(Y, N) = 1$ *then for all* $Y' \geq Y$ *we have* $C(Y', N) = 1$.
2. *if* $C(Y, N) = 1$ *then for all* $N' \leq N$ *we have* $C(Y, N') = 1$.

The majority axiom will be applied in the upcoming characterization results. It is, however, already implied by anonymity, neutrality, and sincere strategy-proofness.

Lemma 3 *(Majority) Axioms A, N, and SP* together imply M.*

5 Symmetric quorum rules

Turning from desirable properties to concrete referendum rules, the following three rules will be relevant in what follows. To begin with, we define the simple majority rule M that returns the majority opinion of the electorate if and only if there is such a majority opinion:

$$M(V) = \begin{cases} 1 & \text{if } n^+(V) > n^-(V) \\ -1 & \text{if } n^-(V) > n^+(V) \\ 0 & \text{otherwise} \end{cases}$$

For many referenda, the validity of the referendum is dependent on participation reaching a certain minimum level. The **participation quorum rule** P_q returns the majority opinion whenever the number of voters reaches the quorum q. Formally, we have:

$$P_q(V) = \begin{cases} 1 & \text{if } n^+(V) > n^-(V) \text{ and } n^+(V) + n^-(V) \geq q \\ -1 & \text{if } n^-(V) > n^+(V) \text{ and } n^+(V) + n^-(V) \geq q \\ 0 & \text{otherwise} \end{cases}$$

Note that the standard majority rule can be obtained as a special case by taking the quorum to be 0 or 1, i.e., we have $M(V) = P_0(V) = P_1(V)$ for all $V \in \mathcal{V}$. Also, P_q is not strategy-proof, this is precisely the content of the no-show paradox. Consider the case of five voters with $q = 3$ and suppose that voter 3 has true preference $\geq_1$. We consider two different ballot situations: In the first situation $(1, 0, x, 0, -1)$, voter 3 can vote $x = 1$ and thereby obtain referendum outcome 1, her most preferred outcome. In this situation abstaining with $x = 0$ would have yielded an invalid referendum outcome 0. In the second situation $(0, 0, x, -1, -1)$, however, a vote $x = 1$ yields referendum outcome -1, her least preferred outcome, whereas voting $x = 0$ yields at least an invalid referendum 0. Hence, the referendum rule P_q has in general no dominant strategy for an individual with preference $\geq_1$, since abstention can be a strategically advantageous choice. The participation quorum rule is depicted in Fig. 1 and has been characterized by Houy (2009).

Finally, we can also consider a different kind of quorum rule, the **majority quorum rule**. Instead of applying the quorum requirement to the set of voters, we can also decide to apply it to the voters who support the majority opinion. In other words, the majority opinion has to be supported by a certain percentage of the population. This is captured by the following referendum rule M_q, one of the two referendum rules we will characterize in this paper:

$$M_q(V) = \begin{cases} 1 & \text{if } n^+(V) > n^-(V) \text{ and } n^+(V) \geq q \\ -1 & \text{if } n^-(V) > n^+(V) \text{ and } n^-(V) \geq q \\ 0 & \text{otherwise} \end{cases}$$

Note that again the standard majority rule can be obtained as a special case where the quorum is 0 or 1, i.e., we have $M(V) = M_0(V) = M_1(V)$ for all $V \in \mathcal{V}$.

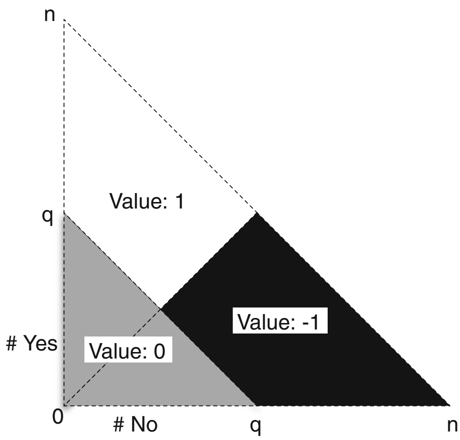


Fig. 1 The participation quorum rule P_q

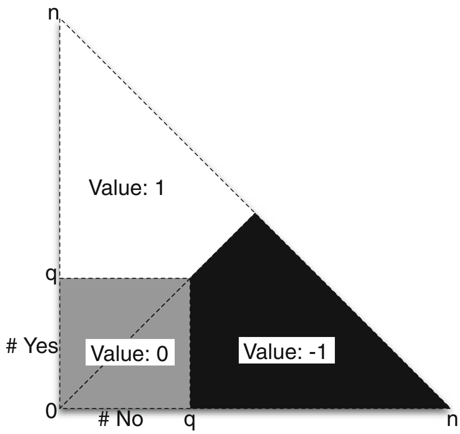


Fig. 2 The majority quorum rule M_q

This referendum rule is the natural symmetric generalization of the acceptance quorum rule to be considered later. The majority quorum rule is depicted in Fig. 2.

The following theorem presents one of the two main contributions of this paper. It can be viewed either as a characterization of the class of majority quorum rules, or as a characterization of the class of minimally non-manipulable referenda with quorums.

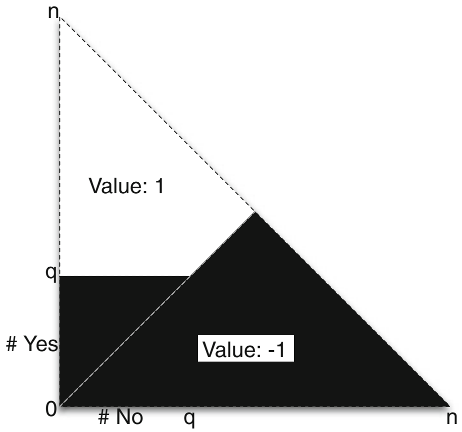
Theorem 1 *(Characterization majority quorum) A referendum rule C satisfies the axioms A, N, WP, Q, and SP if and only if there is a quorum* $q \leq n$ *such that* $C = M_q$.

The axioms A, N, WP, Q, and SP are independent, as is proved in the appendix.

6 Asymmetric quorum rules

As mentioned in the introduction, in many real-life referenda, there is a status quo which a referendum initiative aims to change. In our model, we will identify the status quo with -1, the no option. A referendum with too little participation will then lead

Fig. 3 The acceptance quorum rule A_q



to maintaining the status quo, there is no invalid referendum. Changing the status quo usually requires not just a majority for change, but also a certain quorum of people who vote for change. On the other hand, maintaining the status quo requires no quorum, it is the default option. Hence, there is no symmetry for the two options, the yes option (i.e., change) faces a higher hurdle than the no option (i.e., the status quo).

The formal model of Sect. 4 can remain unchanged. We formally leave the set of possible referendum outcomes as $\{-1, 0, 1\}$ and simply take 0 as a referendum outcome that is never realized. As for the two types of voters considered earlier, we can then also leave the preference relations $\geq_1$ and $\geq_{-1}$ as defined over these three outcomes, knowing that the relevant part of these preferences refers to outcomes 1 and -1 only.

The **acceptance quorum rule** A_q is the asymmetric analogue of the majority quorum rule M_q, and is formalized as follows:

$$A_q(V) = \begin{cases} 1 & \text{if } n^+(V) > n^-(V) \text{ and } n^+(V) \geq q \\ -1 & \text{otherwise} \end{cases}$$

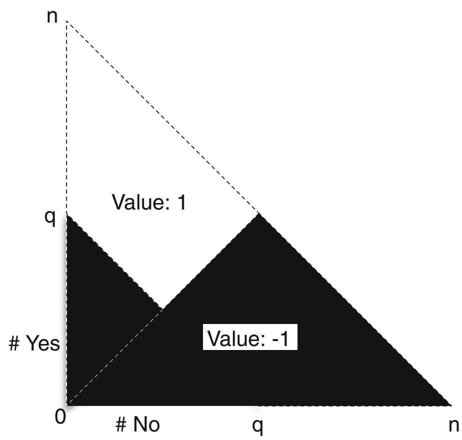
The acceptance quorum rule is depicted in Fig. 3. It is immediately visible that this rule is asymmetric, it does not satisfy the neutrality axiom. Furthermore, it is decisive, i.e., it never returns 0 as a result.

There is also the following **asymmetric participation quorum rule** which we will denote by P'. This rule is depicted in Fig. 4. I am not aware of any characterization result for P'_q.

$$P'_q(V) = \begin{cases} 1 & \text{if } n^+(V) > n^-(V) \text{ and } n^+(V) + n^-(V) \geq q \\ -1 & \text{otherwise} \end{cases}$$

Just like in the case of the symmetric cousin of the acceptance quorum rule, we will now proceed to characterize this referendum rule. We will draw on our previous axioms and also use the following two new axioms. The first axiom RQ translates the quorum

Fig. 4 The asymmetric participation quorum rule P'_q



axiom into the context where an invalid referendum means the same as maintaining the status quo alternative: if there is a majority for change but the referendum result is the status quo, this must mean that there are too few votes, i.e., the quorum has not been met, and hence increasing the number of abstainers should not change the referendum result. The second axiom D formalizes the new restricted referendum context where every referendum outcome is decisive.

Axiom 8 *(RQ, revised quorum) A referendum rule C satisfies the revised quorum condition iff for all* $V \in \mathcal{V}$*, if* $C(V) = -1$ *and* $n^+(V) > n^-(V)$ *then for all* V' *such that* $n^+(V') \le n^+(V)$ *and* $n^-(V') \le n^-(V)$ *we have* $C(V') = -1$.

Axiom 9 *(D, decisiveness) A referendum rule C satisfies decisiveness iff for all* $V \in \mathcal{V}$*, we have* $C(V) = 1$ *or* $C(V) = -1$.

The following result as well as its proof are analogous to the characterization result of symmetric quorum rules, Theorem 1. The differences are worth pointing out in detail. First, the neutrality axiom is absent in Theorem 2, since we are considering asymmetric quorum rules which are by definition not neutral. Second, the decisiveness axiom D is added, and third the quorum axiom Q has been replaced by its revised version RQ. Fourth, the majority axiom needs to be included explicitly in our axiomatization since it is not implied by the remaining axioms (in contrast to Lemma 3 for the symmetric case). To see this, consider the constant referendum rule C defined as $C(V) = 1$ for all $V.C$ satisfies axioms A, WP, RQ, D, and SP* without satisfying the majority axiom M. Fifth, and finally, axiom SP has been replaced by axiom SP*. This is because in the asymmetric case, axiom SP* is not implied by the remaining axiom plus SP. To see this consider the following referendum rule C':

$$C'(V) = \begin{cases} \;\;1 \text{ if } n^+(V) > n^-(V) + n^0(V) \text{ and } n^+(V) + n^-(V) \ge q \\ -1 \text{ otherwise} \end{cases}$$

This referendum rule C' is almost like the revised participation quorum rule P'_q, except that abstentions are also counted as no-votes in determining the referendum outcome.

It can be checked that C' satisfies axioms A, M, WP, D, and RQ. Note also that C' satisfies minimal strategy-proofness, axiom SP: voting yes is a dominant strategy for $\geq_1$ and abstaining is a dominant strategy for $\geq_{-1}$. On the other hand, C' does not satisfy SP* because voting no is not a dominant strategy for $\geq_{-1}$.

Theorem 2 *(Characterization acceptance quorum) A decision rule C satisfies the axioms A, M, WP, RQ, D, and SP* if and only if there is a quorum* $q \leq n$ *such that* $C = A_q$.

Again, the axioms A, M, WP, RQ, D, and SP* are independent, and a proof of this result is contained in the appendix.

7 Conclusions

The main theorem, theorem 1, characterizes majority quorum rules in terms of a strategy-proofness condition that refers to a restricted domain of two particular preference relations. First, it should be noted that by the Gibbard–Satterthwaite impossibility theorem (Gibbard 1973; Satterthwaite 1975), since referendum rules are non-trivial game forms, we cannot hope to obtain a referendum rule which is strategy-proof for all preference relations. In particular, consider the voter considered earlier who mainly cares about participation in direct democracy and hence prefers any valid referendum over an invalid one. If she prefers a yes outcome to a no outcome to an invalid referendum, and if the no-side is in the majority but just fails to reach the quorum, then voting -1 yields outcome -1 and thus is better than voting 1 which would yield outcome 0. Hence, for this voter the optimal ballot depends on the ballots of the other voters, and hence under the majority quorum rule she has no dominant strategy.

Turning from the symmetric to the asymmetric case, note that the Gibbard–Satterthwaite impossibility result does not apply anymore, since here we are essentially in a world of just two alternatives, a world where non-trivial strategy-proof social choice is possible. In line with this fact, we observed that the acceptance quorum rule is not the only rule which is strategy-proof for our two special types of voting preferences. However, Theorem 2 established that it is the only one where sincere voting is a dominant strategy. So whereas for the symmetric case, sincere strategy-proofness and strategy-proofness coincide (see Lemma 1), for asymmetric quorum rules, we saw that strategy-proofness and sincere strategy-proofness diverge (see the referendum rule C' defined in Sect. 6). A more general discussion of (group-)strategy-proofnees for rules with a range of just two elements has been provided by Barberà et al. (2011).

At the end, we return to our original question: which referendum procedures are minimally strategy-proof, i.e., non-manipulable for at least the two kinds of voters we considered? Of course, the answer depends on what we mean by a referendum rule. The axioms of anonymity, neutrality, and weak Pareto optimality should be uncontroversial for any decision rule, at least in the context where the two alternatives are treated symmetrically. The acceptability of the quorum axiom (or the revised quorum axiom in situations where there are only two referendum outcomes) is thus central to our interpretation of the characterization results. Since the axioms of the two main theorems are independent, there are decision rules which satisfy all the axioms except the (revised) quorum axiom. One could certainly imagine contexts in which these rules might seem intuitively reasonable, even though we have not come across any examples of these rules in actual referendum practice.

Accepting the (revised) quorum axiom as part of our notion of a referendum rule, one can interpret the characterization results of this paper in three ways: first, the results can be seen as characterization results for certain referendum rules in terms of strategy-proofness properties. Second, the results can be viewed as characterizing the class of minimally manipulable referendum rules. Third, there is also a practical reading of the results, in that they provide an additional argument for referendum procedures which work with majority or acceptance quorums.

We conclude by placing these results into context and pointing out some avenues for future research. Starting with May's characterization of the majority rule, there now have now been two generalizations of the majority rule, on the one hand to the class of participation quorum rules, and on the other hand to the class of majority quorum rules. Houy (2009) characterized the first class, in this paper we have characterized the second class. Of course, one can always try to obtain alternative characterizations. Furthermore, one could further enlarge the classes of referenda already characterized. In our case, this could mean (1) characterizing asymmetric quorum rules with axiom SP instead of SP*, and (2) characterizing referenda without quorums, i.e., without using the axiom Q (or RQ). The question whether these larger classes allow for equally intuitive and practically relevant characterizations as the quorum rules considered here remains to be investigated.

8 Appendix

Proof of Lemma 1 By axiom SP, we know that individual 1 has a dominant strategy v_* under preference relation $\geq_1$. First note that by axiom A, v_* must be a dominant strategy for all individuals under $\geq_1$. If $v_* = 1$, we are done. So consider the other two cases. Suppose first that $v_* = 0$. Then using axiom WP and the assumption that 0 is a dominant strategy, we have $C(0, 1, \ldots, 1) = 1$. Now we can apply for every individual one by one that 0 is a dominant strategy and obtain $C(0, 0, \ldots, 0) = 1$. Using neutrality, we also have $C(0, 0, \ldots, 0) = -1$, a contradiction. Now suppose second that $v_* = -1$. Then using axiom WP and the assumption that -1 is a dominant strategy, we have $C(-1, 1, \ldots, 1) = 1$, and repeated application for every individual gives us $C(-1, -1, \ldots, -1) = 1$. By axiom N, $C(1, 1, \ldots, 1) = -1$, contradicting WP.

Finally, we also know that individual 1 has a dominant strategy v_* under preference relation $\geq_{-1}$. By axioms N and WP, $C(-1, -1, \ldots, -1) = -1$. The rest of the argument is analogous to the argument for $\geq_1$. □

Proof of Lemma 2 (1.) Using axiom SP*, we have $C(Y+1, N) \geq C(Y, N) = 1$, so $C(Y+1, N) = 1$. Repeatedly applying this argument yields the desired conclusion. (2.) Assuming that $N > 0$, using SP*, we have $1 = C(Y, N) \leq C(Y, N-1)$. Hence, $C(Y, N-1) = 1$. Repeatedly applying this argument yields the desired conclusion. □

Proof of Lemma 3 Since C satisfies A, for any $V \in \mathcal{V}$ we can write $C(V)$ as $C(Y, N)$ with $Y = n^+(V)$ and $N = n^-(V)$. Suppose that $C(Y, N) = 1$. For a proof by contradiction, suppose that $Y \leq N$. We distinguish two cases. First, suppose that $Y = N$. Then by neutrality, $C(N, Y) = -1$. But since $C(N, Y) = C(Y, N)$, we have obtained a contradiction. Second, suppose that $Y < N$. By Lemma 2, $C(Y, Y) = 1$. But then by neutrality again, $C(Y, Y) = -1$ as well, a contradiction. □

Proof of Theorem 1 It is easy to check that any M_q rule satisfies the axioms, so we will only prove the converse. As a visual aid to the proof, it may be useful to construct the diagram of C step by step. Using the axioms we will see that the diagram must look like Fig. 2.

Suppose some decision rule C satisfies the axioms. Given anonymity, we can write C as $C(n^+(V), n^-(V))$. Now let q be the smallest number such that $C(q, r) = 1$ for some r. By WP we know that such a q exists. By axiom M (implied by Lemmas 1 and 3), we know that $q > r$. We will show below that $C = M_q$.

By Lemma 2, we know that for all $Y \geq q$, we have $C(Y, r) = 1$. Again, by Lemma 2, we know that for all $Y \geq q$ and $N \leq r$, we have $C(Y, N) = 1$.

Now suppose we have $Y \geq q$ and $N \geq r$ with $Y > N$. By axioms M and N, we know that $C(Y, N) \neq -1$. Also, by Q, $C(Y, N) \neq 0$, otherwise we would have $C(q, r) = 0$. Hence, we know that as long as $Y > N$, $C(Y, N) = 1$ for all $Y \geq q$. Note that we have just proved the first clause of the definition of M_q.

To show the second clause of the definition of M_q, suppose that $N > Y$ and that $N \geq q$. Then by neutrality, $C(Y, N) = -C(N, Y) = -1$.

Finally, as for the third clause in the definition of M_q, we distinguish three cases. First, suppose that $Y = N$. Then by axioms M and N, we know that $C(Y, N) = 0$. Second, consider that $Y > N$ but that $Y < q$. We know that $C(Y, N) \neq 1$ since q was taken to be the lowest point for which C yields 1. Also, we know that $C(Y, N) \neq -1$ for that would contradict the majority axiom M using neutrality. Hence, $C(Y, N) = 0$. The third case where $Y < N$ with $N < q$ follows by neutrality from this second case: $C(Y, N) = -C(N, Y) = 0$.

Since we have shown that in all cases C yields the same result as M_q, we have completed the proof. □

Proof of the independence of axioms A, N, WP, Q, and SP:

NOT SP We have already seen earlier that the participation quorum rule P_q is not strategy-proof. It can be checked that it satisfies all the other axioms.

NOT WP Consider the referendum rule that constantly returns 0 on all inputs, i.e., $C(V) = 0$ for all $V \in \mathcal{V}$. It satisfies all the axioms except WP.

NOT N Consider the acceptance quorum rule A_q, defined in Sect. 6 and depicted in Fig. 3. It satisfies all axioms except neutrality (see also Theorem 2).

NOT Q Let C be defined just like M_q, except that for all $X > 0$ we have $C(X, 0) = 1$ and $C(0, X) = -1$. Intuitively, this referendum rule yields the majority opinion if the quorum is met, but if nobody opposes the majority opinion, the majority opinion results even if the quorum is not met. C does not satisfy Q since for $q = 3$, we have $C(1, 2) = 0$ while $C(1, 0) = 1$. However, C satisfies the other axioms. To see that C satisfies SP* (and hence also SP), we only show part (1.) of the axiom since part (2.) can be shown analogously. For part (1.), it suffices to verify that $C(V[i : 1]) \geq C(V)$ for the cases where one of the two sides of the inequality is $C(X, 0)$ or $C(0, X)$. If $C(X, 0)$ occurs on the left side of the inequality or $C(0, X)$ occurs on the right side, the inequality is automatically satisfied. Furthermore, $C(0, X)$ cannot occur on the left side. Hence, we are left with the case that $C(X, 0)$ occurs on the right side. In this case, we need to check that $C(X + 1, 0) = C(X, 0) = 1$ which holds by definition of C.

NOT A Consider the referendum rule C^n which uses a majority quorum but counts voter 1 twice. Formally, we have $C^n(v_1, \ldots, v_n) = M_q^{n+1}(v_1, \ldots, v_n, v_1)$. This rule is clearly not anonymous. Neutrality and WP are easily seen to be satisfied by C. To show that C satisfies SP*, we use twice the fact that M_q^{n+1} satisfies SP*:

$$\begin{aligned} C(1, v_2, \ldots, v_n) &= M_q^{n+1}(1, v_2, \ldots, v_n, 1) \\ &\geq M_q^{n+1}(k, v_2, \ldots, v_n, 1) \\ &\geq M_q^{n+1}(k, v_2, \ldots, v_n, k) = C(k, v_2, \ldots, v_n) \end{aligned}$$

Finally, we note that C also satisfies Q: M_q^{n+1} satisfies Q over the set of all vote vectors, so in particular it satisfies Q over the restricted set of vote vectors where the first and the last voter submit identical votes. □

Proof of Theorem 2 It is easy to check that any A_q rule satisfies the axioms, so we will only prove the converse.

Suppose some decision rule C satisfies the axioms. Given anonymity, we can write C as $C(n^+(V), n^-(V))$. Now let q be the smallest number such that $C(q, r) = 1$ for some r. By WP we know that such a q exists. By axiom M, we know that $q > r$. We will show below that $C = A_q$.

By Lemma 2, we know that for all $Y \geq q$, we have $C(Y, r) = 1$. Again, by Lemma 2, we know that for all $Y \geq q$ and $N \leq r$, we have $C(Y, N) = 1$.

Now suppose we have $Y \geq q$ and $N \geq r$ with $Y > N$. By RQ, we know that $C(Y, N) \neq -1$ (otherwise we would have $C(q, r) = -1$ as well), and so by axiom D, $C(Y, N) = 1$. Hence, we know that as long as $Y > N$, $C(Y, N) = 1$ for all $Y \geq q$, proving the first clause of the definition of A_q.

For the second clause of the definition, we have two cases. First, suppose that $Y \leq N$. Then by axioms M and D, $C(Y, N) = -1$. Second, suppose that $Y < q$. Then we know that $C(Y, N) = -1$ by axiom D and by our assumption that q was the smallest value for C to yield 1. □

Proof of the independence of axioms A, M, WP, RQ, D, and SP:*

NOT M As was already mentioned, we can consider the constant referendum rule $C(V) = 1$ for all V.

NOT D The majority quorum rule M_q does not satisfy decisiveness but does satisfy RQ (and all the other axioms).

NOT WP Consider the referendum rule that constantly returns -1 on all inputs. It violates WP but satisfies all the other axioms.

*NOT SP** The participation quorum rule P' satisfies all the axioms except strategy-proofness. Consider the case of 5 voters with $q = 3$, where voter 3 has preference $\geq_{-1}$, and the following ballot profile $V = (1, 1, -1, 0, 0)$. When submitted, we have $P'_q(V) = 1$, the opposite of what voter 3 wants. However, voter 3 can behave strategically by abstaining. We then obtain $P'_q(V[3 : 0]) = -1$. Hence, $P'_q(V) \not\leq P'_q(V[3 : 0])$, a violation of axiom SP*.

NOT A Define $C(k, 1, \ldots, 1) = 1$ for all $k \in \{-1, 0, 1\}$ and $(V) = -1$ otherwise. This rule is not anonymous, but all the other axioms are satisfied by C.

NOT RQ Let $C(V) = 1$ iff $n^+(V) > n^-(V), n^+(V) \geq q$ and $n^-(V) \leq q$. This referendum rule adds to the acceptance quorum rule an additional requirement for acceptance, namely that there are at most q no-votes. It does not satisfy RQ, since $C(q + 2, q + 1) = -1$ while $C(q + 2, q) = 1$. Axioms A, M, D, and WP are clearly satisfied. For SP*, first we must show that $C(V[i : 1]) \geq C(V)$ for all V. If $C(V) = 1$, all three conjuncts in the definition of C are true, and they will remain true when increasing i's ballot. Second, we need to show that $C(V) \geq C(V[i : -1])$. If $C(V) = -1$, one of the three conjuncts in the defining clause of C must be false, and lowering i's ballot will not change this fact. □

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