World Input-Output Database

Fragmentation in an Inter-country Input-Output Framework

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Abstract
The import content of the exports of a country has been widely used as a measure for vertical specialization. Inter-country input-output tables reflect the trade dependencies between countries much better than national tables do. This paper rigorously proves, however, that the vertical specialization is exactly the same for a national and an inter-country input-output table. This result implies that for a comparison across countries, the laborious task of estimating inter-country input-output tables is no longer necessary.

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1. Introduction

Fragmentation was introduced by Jones and Kierzkowski (1990, 2005) to describe the organization of production processes. More and more, production processes are split into subsequent phases, which are carried out separately and in different countries. As a consequence, the trade of intermediate products becomes more important and vertical trading chains exhibit an increasing interconnectedness of industries across countries. Vertical specialization occurs when each country specializes in certain stages of the sequence of production. In an influential paper, Hummels *et al.* (2001) narrowed the concept of vertical specialization by focusing on the imported inputs that are necessary for producing the exports. Vertical specialization for a country was measured by the export weighted average direct import coefficient or by the export weighted average import multiplier (including also the indirect import requirements). Essentially, what this measures is the import content of the exports. The empirical results were obtained from applying these measures to national input-output tables.

The present paper extends the analysis by taking inter-country input-output tables as the starting-point. These tables are of the interregional type with countries acting as regions. Inter-country tables provide a detailed description of the interdependencies of industries between countries and thus reflect exactly what we are interested in. It may thus be expected that the results obtained from inter-country tables measure vertical specialization better than national tables do.

In this paper, it is rigorously proven that this is not the case. Distinguishing between intra-country effects, inter-country spillovers and inter-country feedback effects it is shown that vertical specialization when measured with an inter-country table is the same as when measured with a national table.

This result has far-reaching consequences because inter-country input-output tables are not readily available and need to be constructed (which is a painstaking and time-consuming process). In contrast, national input-output tables are now widely available and do not require much additional work if one aims at a comparison of national vertical specialization across countries.

2. Measures for vertical specialization

The starting-point is the inter-country input-output table in Figure 1. We distinguish between countries 1 and 2, and the rest of the world R. The elements z_{ij}^{11} of matrix $\mathbf{Z}^{11}$ give the domestic (or intra-country) intermediate deliveries of country 1, i.e. the deliveries of products from industry i in country 1 for input use in industry j in country 1, with $i, j = 1, \ldots, n$, where n is the number of industries. In the same way, the elements z_{ij}^{12} of matrix $\mathbf{Z}^{12}$ give the deliveries of products from industry i in country 1 for input use in industry j in country 2. The interpretation of matrices $\mathbf{Z}^{21}$ and $\mathbf{Z}^{22}$ is similar. Note that country 2 may actually be a group of (say k) countries, in which case the dimensions of the matrices changes correspondingly. For example, $\mathbf{Z}^{12}$ is—in that case—no longer an $n\times n$ matrix but becomes $n\times kn$.

Figure 1. The inter-country input-output table

$\mathbf{Z}^{11}$	$\mathbf{Z}^{12}$	$\mathbf{c}^{1}$	$\mathbf{e}^{1}$	$\mathbf{x}^{1}$
$\mathbf{Z}^{21}$	$\mathbf{Z}^{22}$	$\mathbf{c}^{2}$	$\mathbf{e}^{2}$	$\mathbf{x}^{2}$
$\mathbf{Z}^{R1}$	$\mathbf{Z}^{RR}$			
$(\mathbf{v}^{1})'$	$(\mathbf{v}^{2})'$			
$(\mathbf{x}^{1})'$	$(\mathbf{x}^{2})'$			

The $n\times 1$ (column) vector $\mathbf{c}^{1}$ indicates the domestic final demand (private consumption, private investments, and government expenditures) for products from industries in country 1. The $n\times 1$ (column) vector $\mathbf{e}^{1}$ gives the exports to ‘consumers’ (i.e. consumers, investors, and government) in country 2 and to the rest of the world. The elements of the $n \times n$ matrix $\mathbf{Z}^{R1}$ indicate the imports from the rest of the world by industries in country 1. The $1 \times n$ (row) vector $(\mathbf{v}^1)'$ gives the value added in each industry in country 1. Gross outputs in country 1 are given by the elements of the vector $\mathbf{x}^1$.

Direct input coefficients are obtained from normalizing the industry columns in the IO table. For country 1 this yields $\mathbf{A}^{11} = \mathbf{Z}^{11}(\hat{\mathbf{x}}^1)^{-1}$ for the domestically produced inputs, $\mathbf{A}^{21} = \mathbf{Z}^{21}(\hat{\mathbf{x}}^1)^{-1}$ for the inputs imported from country 2, and $\mathbf{A}^{R1} = \mathbf{Z}^{R1}(\hat{\mathbf{x}}^1)^{-1}$ for the inputs imported from the rest of the world. For instance, element a_{ij}^{21} gives the input (say in dollars) of product *i* from country 2 used per unit (i.e. dollar) of output of product *j* in country 1.

2.1. The single-country framework

In the single-country case as introduced by Hummels *et al*. (2001), the question is how much imports are required for the exports. The total amount of exports of country 1 is given by $\mathbf{Z}^{12}\mathbf{s} + \mathbf{e}^1$, where $\mathbf{s}$ indicates the summation vector consisting entirely of ones. The first part gives the exports of country 1 that are used in country 2 in their production process. The second part gives the exports of country 1 to country 2 to its 'consumers' (i.e. exports that are not used as inputs into the production process) and all exports to the rest of the world. For convenience, write

$$\mathbf{e} = \mathbf{Z}^{12}\mathbf{s} + \mathbf{e}^1. \qquad (1)$$

The domestic outputs necessary for the exports vector $\mathbf{e}$ are given by $(\mathbf{I} - \mathbf{A}^{11})^{-1}\mathbf{e}$. Producing outputs to the amount of $\mathbf{x}^1$ requires imports $\mathbf{M}\mathbf{x}^1$, where $\mathbf{M}$ gives the matrix of all direct import coefficients irrespective of their origin, i.e. $\mathbf{M} = \mathbf{A}^{21} + \mathbf{A}^{R1}$. Hence, the imports necessary for the exports $\mathbf{e}$ are given by $\mathbf{M}(\mathbf{I} - \mathbf{A}^{11})^{-1}\mathbf{e}$ and the total imports are obtained from summation which yields $\mathbf{s}'\mathbf{M}(\mathbf{I} - \mathbf{A}^{11})^{-1}\mathbf{e}$.

Note that the row vector

$$\boldsymbol{\mu}' = \mathbf{s}'\mathbf{M}(\mathbf{I} - \mathbf{A}^{11})^{-1} \tag{2}$$

gives the import multipliers. For example, μ_j gives the total amount of imports that is directly and indirectly required to satisfy one unit (or dollar) of final demand—for example export—of product *j*. The measure for vertical specialization proposed by Hummels *et al*. (2001) used the import content of the exports and is given by

$$VS^{single} = \frac{\mathbf{s}'\mathbf{M}(\mathbf{I} - \mathbf{A}^{11})^{-1}\mathbf{e}}{\mathbf{s}'\mathbf{e}} = \frac{\boldsymbol{\mu}'\mathbf{e}}{\mathbf{s}'\mathbf{e}} = \frac{\Sigma_i \mu_i e_i}{\Sigma_i e_i} \tag{3}$$

The degree of vertical specialization is thus obtained as the weighted average of the import multipliers, using the sectoral exports as weights.

2.2. The inter-country framework

In the full inter-country setting, the situation is less straightforward. First, because inter-country spillover and feedback effects will occur. The exports $\mathbf{e}^1$ for example, require production in country 1, which in its turn requires imported inputs from—and thus production in—country 2. This is an inter-country spillover. As a consequence, however, this production in country 2 requires imported inputs from—and thus production in—country 1. This is an inter-country feedback effect. Second, the exports of country 1 are no longer exogenously specified (as was the case in the single-country case). In the present case, they are endogenously determined by the gross outputs in country 2, which in themselves are endogenously determined, and so forth.

One of characteristics of the traditional IO model is that the final demands are given exogenously and that gross outputs are determined endogenously. Once the outputs are known all sorts of 'appended' requirements (such as imports from the rest of the world) can be determined. Taking the IO table from Figure 1 as the starting-point, there are three sources of exogenous final demands: $\mathbf{c}^1$, $\mathbf{e}^1$, and $\mathbf{c}^2 + \mathbf{e}^2$ (it is not necessary for the present analysis to split the final demands in country 2). In what follows, I will determine the exports and the imports generated by each of these three final demand components.

Let us start with the export vector $\mathbf{e}^1$. Due to intra-country effects these exports require production in country 1 to the amount of $(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{e}^1$. For this production, however, imported inputs are required: $\mathbf{A}^{21}(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{e}^1$ from country 2 and $\mathbf{A}^{R1}(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{e}^1$ from the rest of the world. Let us write $\mathbf{K}^{21}=\mathbf{A}^{21}(\mathbf{I}-\mathbf{A}^{11})^{-1}$ and $\mathbf{K}^{R1}=\mathbf{A}^{R1}(\mathbf{I}-\mathbf{A}^{11})^{-1}$. The imports $\mathbf{K}^{21}\mathbf{e}^1$ from country 2 (which is an inter-country spillover effect) induce production in country 2 from intra-country effects, i.e. $(\mathbf{I}-\mathbf{A}^{22})^{-1}\mathbf{K}^{21}\mathbf{e}^1$. This production requires imported inputs from country 1 (which is an inter-country feedback effect) to the amount of $\mathbf{A}^{12}(\mathbf{I}-\mathbf{A}^{22})^{-1}\mathbf{K}^{21}\mathbf{e}^1$. Writing $\mathbf{K}^{12}=\mathbf{A}^{12}(\mathbf{I}-\mathbf{A}^{22})^{-1}$ yields $\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{e}^1$ for these imports of country 2 (which are exports of country 1). In its turn, these exports imply production in country 1 and consequently imported inputs. These imports are $\mathbf{K}^{21}\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{e}^1$ from country 2 and $\mathbf{K}^{R1}\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{e}^1$ from the rest of the world. The imports from country 2 yield production in country 2 which requires that country 1 exports inputs to the amount of $\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{e}^1$, and so forth.

Collecting terms yields for the exports of country 1

$$\mathbf{e}^1+\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{e}^1+\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{e}^1+\ldots=(\mathbf{I}-\mathbf{K}^{12}\mathbf{K}^{21})^{-1}\mathbf{e}^1 \tag{4}$$

Collecting term for the imports (from country 2 and from the rest of the world) yields

$$\begin{aligned}&(\mathbf{K}^{21}+\mathbf{K}^{R1})\mathbf{e}^1+(\mathbf{K}^{21}+\mathbf{K}^{R1})\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{e}^1+\ldots=\\&\quad(\mathbf{K}^{21}+\mathbf{K}^{R1})(\mathbf{I}-\mathbf{K}^{12}\mathbf{K}^{21})^{-1}\mathbf{e}^1\end{aligned} \tag{5}$$

Next consider the domestic final demands $\mathbf{c}^1$. They imply production in country 1, given by $(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{c}^1$, which requires imports. These are imports from country 2, i.e. $\mathbf{A}^{21}(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{c}^1=\mathbf{K}^{21}\mathbf{c}^1$, and imports from the rest of the world, i.e. $\mathbf{A}^{R1}(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{c}^1=\mathbf{K}^{R1}\mathbf{c}^1$. These imports from country 2 imply production in country 2 and, thus, imported inputs from country 1. These exports by country 1 are given by $\mathbf{A}^{12}(\mathbf{I}-\mathbf{A}^{22})^{-1}\mathbf{K}^{21}\mathbf{c}^{1}=\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{c}^{1}$. The exports imply further production and imports ($\mathbf{K}^{21}\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{c}^{1}$ and $\mathbf{K}^{R1}\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{c}^{1}$), and so forth. Collecting terms for the exports gives

$$\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{c}^{1}+\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{c}^{1}+\ldots=(\mathbf{I}-\mathbf{K}^{12}\mathbf{K}^{21})^{-1}\mathbf{c}^{1}-\mathbf{c}^{1} \tag{6}$$

Note that the "first" imports above, i.e. $\mathbf{K}^{21}\mathbf{c}^{1}+\mathbf{K}^{R1}\mathbf{c}^{1}$, are not generated by exports (but by domestic 'consumption' in country 1) and are therefore not included when calculating the import contents of the exports. Hence, collecting the remaining imports yields

$$\begin{aligned}&(\mathbf{K}^{21}+\mathbf{K}^{R1})\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{c}^{1}+(\mathbf{K}^{21}+\mathbf{K}^{R1})\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{c}^{1}+\ldots=\\&\quad(\mathbf{K}^{21}+\mathbf{K}^{R1})[(\mathbf{I}-\mathbf{K}^{12}\mathbf{K}^{21})^{-1}\mathbf{c}^{1}-\mathbf{c}^{1}]\end{aligned} \tag{7}$$

The last part in determining the import content of the exports, considers the exports (and the consequent imports) due to the final demands in country 2. Starting-point is the final demand vector $\mathbf{c}^{2}+\mathbf{e}^{2}$, which requires production in country 2 equal to $(\mathbf{I}-\mathbf{A}^{22})^{-1}(\mathbf{c}^{2}+\mathbf{e}^{2})$. In its turn, imported inputs from country 1 are required to the amount of $\mathbf{A}^{12}(\mathbf{I}-\mathbf{A}^{22})^{-1}(\mathbf{c}^{2}+\mathbf{e}^{2})=\mathbf{K}^{12}(\mathbf{c}^{2}+\mathbf{e}^{2})$. These exports of country 1 imply production $(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{K}^{12}(\mathbf{c}^{2}+\mathbf{e}^{2})$, which requires imported inputs from country 2, i.e. $\mathbf{A}^{21}(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{K}^{12}(\mathbf{c}^{2}+\mathbf{e}^{2})=\mathbf{K}^{21}\mathbf{K}^{12}(\mathbf{c}^{2}+\mathbf{e}^{2})$, and from the rest of the world, i.e. $\mathbf{A}^{R1}(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{K}^{12}(\mathbf{c}^{2}+\mathbf{e}^{2})=\mathbf{K}^{R1}\mathbf{K}^{12}(\mathbf{c}^{2}+\mathbf{e}^{2})$. The imports from country 2 again lead to production in country 2 and import of inputs from country 1, which are given by $\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{K}^{12}(\mathbf{c}^{2}+\mathbf{e}^{2})$, and so forth. Collecting the exports of country 1 gives

$$\mathbf{K}^{12}(\mathbf{c}^{2}+\mathbf{e}^{2})+\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{K}^{12}(\mathbf{c}^{2}+\mathbf{e}^{2})+\ldots=\mathbf{K}^{12}(\mathbf{I}-\mathbf{K}^{21}\mathbf{K}^{12})^{-1}(\mathbf{c}^{2}+\mathbf{e}^{2}) \tag{8}$$

Collecting the imports that were induced by the exports gives

$$(\mathbf{K}^{21}+\mathbf{K}^{R1})\mathbf{K}^{12}(\mathbf{c}^{2}+\mathbf{e}^{2})+(\mathbf{K}^{21}+\mathbf{K}^{R1})\mathbf{K}^{12}\mathbf{K}^{21}\mathbf{K}^{12}(\mathbf{c}^{2}+\mathbf{e}^{2})+\ldots= \\ (\mathbf{K}^{21}+\mathbf{K}^{R1})\mathbf{K}^{12}(\mathbf{I}-\mathbf{K}^{21}\mathbf{K}^{12})^{-1}(\mathbf{c}^{2}+\mathbf{e}^{2}) \tag{9}$$

Summing the exports from equations (4), (6) and (8) gives

$$\mathbf{exp}=(\mathbf{I}-\mathbf{K}^{12}\mathbf{K}^{21})^{-1}(\mathbf{c}^{1}+\mathbf{e}^{1})-\mathbf{c}^{1}+\mathbf{K}^{12}(\mathbf{I}-\mathbf{K}^{21}\mathbf{K}^{12})^{-1}(\mathbf{c}^{2}+\mathbf{e}^{2}) \tag{10}$$

Summing the imports given in equations (5), (7) and (9) gives

$$\mathbf{imp}=(\mathbf{K}^{21}+\mathbf{K}^{R1})[(\mathbf{I}-\mathbf{K}^{12}\mathbf{K}^{21})^{-1}(\mathbf{c}^{1}+\mathbf{e}^{1})-\mathbf{c}^{1}+ \\ \mathbf{K}^{12}(\mathbf{I}-\mathbf{K}^{21}\mathbf{K}^{12})^{-1}(\mathbf{c}^{2}+\mathbf{e}^{2})] \tag{11}$$

The relationship between the imports that are required for the exports is given by

$$\mathbf{imp}=(\mathbf{K}^{21}+\mathbf{K}^{R1})\mathbf{exp}=(\mathbf{A}^{21}+\mathbf{A}^{R1})(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{exp} \tag{12}$$

The next step is to derive the expression for the measure for vertical specialization. The total import content of the exports is given by $\mathbf{s}'\mathbf{imp}$ and the total amount of exports is $\mathbf{s}'\mathbf{exp}$. Note that $\mathbf{s}'\mathbf{imp}=\mathbf{s}'(\mathbf{A}^{21}+\mathbf{A}^{R1})(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{exp}=\boldsymbol{\mu}'\mathbf{exp}$, with $\boldsymbol{\mu}$ the vector of import multipliers defined in (2). Using $\mathbf{M}=\mathbf{A}^{21}+\mathbf{A}^{R1}$, this yields for the inter-country framework

$$VS^{inter}=\frac{\mathbf{s}'\mathbf{M}(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{exp}}{\mathbf{s}'\mathbf{exp}}=\frac{\boldsymbol{\mu}'\mathbf{exp}}{\mathbf{s}'\mathbf{exp}}=\frac{\Sigma_i\mu_i exp_i}{\Sigma_i exp_i} \tag{13}$$

Also in the inter-country setting, vertical specialization is measured as the weighted average of the import multipliers, using the exports as weights. Expression (13) is exactly the same as expression (3), except that the weights may be different. In the next section we will show that this is not the case.

2.3. Completing the proof

In this subsection it will be shown that the exports vector **exp** in (10) is the same as the exports vector in (1) that was used in the single-country case, i.e. $\mathbf{e} = \mathbf{Z}^{12}\mathbf{s} + \mathbf{e}^{1}$. For this, the inter-country input-output model is needed. From the table in Figure 1 and the definition of the input coefficients, it follows that

$$\begin{pmatrix} \mathbf{x}^{1} \\ \mathbf{x}^{2} \end{pmatrix} = \begin{bmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} \\ \mathbf{A}^{21} & \mathbf{A}^{22} \end{bmatrix} \begin{pmatrix} \mathbf{x}^{1} \\ \mathbf{x}^{2} \end{pmatrix} + \begin{pmatrix} \mathbf{c}^{1} + \mathbf{e}^{1} \\ \mathbf{c}^{2} + \mathbf{e}^{2} \end{pmatrix} \tag{14}$$

and its solution is given by

$$\begin{pmatrix} \mathbf{x}^{1} \\ \mathbf{x}^{2} \end{pmatrix} = \begin{bmatrix} \mathbf{I} - \mathbf{A}^{11} & -\mathbf{A}^{12} \\ -\mathbf{A}^{21} & \mathbf{I} - \mathbf{A}^{22} \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{c}^{1} + \mathbf{e}^{1} \\ \mathbf{c}^{2} + \mathbf{e}^{2} \end{pmatrix} = \begin{bmatrix} \mathbf{L}^{11} & \mathbf{L}^{12} \\ \mathbf{L}^{21} & \mathbf{L}^{22} \end{bmatrix} \begin{pmatrix} \mathbf{c}^{1} + \mathbf{e}^{1} \\ \mathbf{c}^{2} + \mathbf{e}^{2} \end{pmatrix} \tag{15}$$

where the inverse matrix is the Leontief inverse. Using well-known expressions for the inverse of a partitioned matrix (see e.g. Miller and Blair, 2009), we have that

$$\mathbf{L}^{11} = [\mathbf{I} - \mathbf{A}^{11} - \mathbf{A}^{12}(\mathbf{I} - \mathbf{A}^{22})^{-1}\mathbf{A}^{21}]^{-1} \tag{16}$$

This can be rewritten as

$$\begin{aligned} \mathbf{L}^{11} &= \{[\mathbf{I} - \mathbf{A}^{12}(\mathbf{I} - \mathbf{A}^{22})^{-1}\mathbf{A}^{21}(\mathbf{I} - \mathbf{A}^{11})^{-1}](\mathbf{I} - \mathbf{A}^{11})\}^{-1} \\ &= (\mathbf{I} - \mathbf{A}^{11})^{-1}[\mathbf{I} - \mathbf{A}^{12}(\mathbf{I} - \mathbf{A}^{22})^{-1}\mathbf{A}^{21}(\mathbf{I} - \mathbf{A}^{11})^{-1}]^{-1} \\ &= (\mathbf{I} - \mathbf{A}^{11})^{-1}[\mathbf{I} - \mathbf{K}^{12}\mathbf{K}^{21}]^{-1} \end{aligned}$$

which implies that

$$(\mathbf{I} - \mathbf{K}^{12}\mathbf{K}^{21})^{-1} = (\mathbf{I} - \mathbf{A}^{11})\mathbf{L}^{11}. \tag{17}$$

Replacing the 1's by 2's, and vice versa, yields $(\mathbf{I}-\mathbf{K}^{21}\mathbf{K}^{12})^{-1}=(\mathbf{I}-\mathbf{A}^{22})\mathbf{L}^{22}$. Pre-multiplying both sides by $\mathbf{K}^{12}=\mathbf{A}^{12}(\mathbf{I}-\mathbf{A}^{22})^{-1}$ gives

$$\mathbf{K}^{12}(\mathbf{I}-\mathbf{K}^{21}\mathbf{K}^{12})^{-1}=\mathbf{A}^{12}(\mathbf{I}-\mathbf{A}^{22})^{-1}(\mathbf{I}-\mathbf{A}^{22})\mathbf{L}^{22}=\mathbf{A}^{12}\mathbf{L}^{22}.$$

From the inverse of a partitioned matrix, it also follows that $\mathbf{L}^{12}=(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{A}^{12}\mathbf{L}^{22}$. This implies that $\mathbf{A}^{12}\mathbf{L}^{22}=(\mathbf{I}-\mathbf{A}^{11})\mathbf{L}^{12}$, so that

$$\mathbf{K}^{12}(\mathbf{I}-\mathbf{K}^{21}\mathbf{K}^{12})^{-1}=(\mathbf{I}-\mathbf{A}^{11})\mathbf{L}^{12}. \tag{18}$$

Substituting expressions (17) and (18) into (10) gives

$$\begin{aligned}\mathbf{exp}&=(\mathbf{I}-\mathbf{K}^{12}\mathbf{K}^{21})^{-1}(\mathbf{c}^1+\mathbf{e}^1)-\mathbf{c}^1+\mathbf{K}^{12}(\mathbf{I}-\mathbf{K}^{21}\mathbf{K}^{12})^{-1}(\mathbf{c}^2+\mathbf{e}^2)\\&=(\mathbf{I}-\mathbf{A}^{11})\mathbf{L}^{11}(\mathbf{c}^1+\mathbf{e}^1)-\mathbf{c}^1+(\mathbf{I}-\mathbf{A}^{11})\mathbf{L}^{12}(\mathbf{c}^2+\mathbf{e}^2)\\&=(\mathbf{I}-\mathbf{A}^{11})[\mathbf{L}^{11}(\mathbf{c}^1+\mathbf{e}^1)+\mathbf{L}^{12}(\mathbf{c}^2+\mathbf{e}^2)]-\mathbf{c}^1\\&=(\mathbf{I}-\mathbf{A}^{11})\mathbf{x}^1-\mathbf{c}^1\end{aligned}$$

where (15) was used. Next observe that it follows from equation (14) that

$$(\mathbf{I}-\mathbf{A}^{11})\mathbf{x}^1-\mathbf{c}^1=\mathbf{A}^{12}\mathbf{x}^2+\mathbf{e}^1=\mathbf{Z}^{12}\mathbf{s}+\mathbf{e}^1=\mathbf{e}$$

where the definition of the input coefficients, i.e. $\mathbf{A}^{12}=\mathbf{Z}^{12}(\hat{\mathbf{x}}^2)^{-1}$, was used.

3. The domestic value added incorporated in the exports

Value added is the complement of the imports and together they completely end up in the final demands. Define $(\mathbf{w}^1)'=(\mathbf{v}^1)'(\hat{\mathbf{x}}^1)^{-1}$ for the vector of value added coefficients, which give the value added in a sector per unit of its output. In the single-country framework we have that the value added directly and indirectly embodied in the exports is given by $(\mathbf{w}^1)'(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{e}$. The vector $\boldsymbol{\lambda}' = (\mathbf{w}^1)'(\mathbf{I}-\mathbf{A}^{11})^{-1}$ gives the value added multipliers, its *i*th element measuring the value added generated per unit of final demand of product *i*. The value added content of the exports is given by

$$VA^{single} = \frac{(\mathbf{w}^1)'(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{e}}{\mathbf{s}'\mathbf{e}} = \frac{\boldsymbol{\lambda}'\mathbf{e}}{\mathbf{s}'\mathbf{e}} = \frac{\Sigma_i \lambda_i e_i}{\Sigma_i e_i}$$

Note that the value added multiplier plus the import multiplier equal one. That is, $\boldsymbol{\lambda}' + \boldsymbol{\mu}' = (\mathbf{w}^1)'(\mathbf{I}-\mathbf{A}^{11})^{-1} + \mathbf{s}'\mathbf{M}(\mathbf{I}-\mathbf{A}^{11})^{-1} = [(\mathbf{w}^1)' + \mathbf{s}'\mathbf{M}](\mathbf{I}-\mathbf{A}^{11})^{-1}$. Note that it follows from the input-output table in Figure 1 that $\mathbf{s}'\mathbf{M} = \mathbf{s}'\mathbf{A}^{21} + \mathbf{s}'\mathbf{A}^{R1} = \mathbf{s}'(\mathbf{I}-\mathbf{A}^{11}) - (\mathbf{w}^1)'$, hence $\boldsymbol{\lambda}' + \boldsymbol{\mu}' = \mathbf{s}'$. Consequently, we have that $VA^{single} = 1 - VS^{single}$.

Similar to (12), we find in the inter-country framework that the value added is given by

$$(\mathbf{w}^1)'(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{exp}$$

with $\mathbf{exp} = [(\mathbf{I}-\mathbf{K}^{12}\mathbf{K}^{21})^{-1}(\mathbf{c}^1 + \mathbf{e}^1) - \mathbf{c}^1 + \mathbf{K}^{12}(\mathbf{I}-\mathbf{K}^{21}\mathbf{K}^{12})^{-1}(\mathbf{c}^2 + \mathbf{e}^2)] = \mathbf{e}$ (from Section 2.3). This implies that

$$VA^{inter} = \frac{(\mathbf{w}^1)'(\mathbf{I}-\mathbf{A}^{11})^{-1}\mathbf{exp}}{\mathbf{s}'\mathbf{exp}} = \frac{\boldsymbol{\lambda}'\mathbf{exp}}{\mathbf{s}'\mathbf{exp}} = \frac{\boldsymbol{\lambda}'\mathbf{e}}{\mathbf{s}'\mathbf{e}} = VA^{single}$$

Moreover, because $\boldsymbol{\lambda}' + \boldsymbol{\mu}' = \mathbf{s}'$, we also have that $VA^{inter} = 1 - VS^{inter}$.

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