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Abstract

The paper presents a generalized dynamic arbitrage free yield model. It is a factor model where the past values of the factors are needed to specify the present yield curve. However, all this past information is contained in the time t yield curve.

Key words: Term structure of interest rates; Arbitrage free yields
JEL classification: E43

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1 Introduction

The premise of a dynamic factor model for yield curves is that a few latent factors $X_{1t}, \dots, X_{mt}$ drive the comovements of yields or bond prices of all maturities. In dynamic factor models, such as arbitrage-free affine yield models (see, for example, Vasicek (1977), Cox, Ingersoll and Ross (1985), and multifactor models as desribed in Duffie and Kan (1996) and Dai and Singleton (2000)), the finite state vector of factors at time t fully specifies the yield curve at time t. In the factor model with infinite dynamics that we propose the factors at time t are not sufficient. Instead past factors $X_{1s}, \dots, X_{ms}$ of all times $s \leq t$ are needed to fully specify the yield curve at time t. However, all this past information is contained in the time t yield curve. As a result today's yield curve and tomorrow's factors contain all information to specify tomorrow's yield curve. We will refer to it as a Generalized Dynamic Arbitrage Free (GD-AF) yield model.

2 A GD-AF yield model

Let the process for the real pricing kernel, or stochastic discount factor, be given by

$$\mathrm{d}\Lambda_t = -\Lambda_t \left(r_t \,\mathrm{d}t + \sum_{j=1}^{m} \lambda_{jt} \,\mathrm{d}W_{jt}^{\mathrm{R}} \right), \tag{1}$$

where r_t is the short rate and $\lambda_{j,t}$, $j = 1, \dots, m$ are prices of risk, related to the independent Brownian motions $W_1^{\mathrm{R}}, \dots, W_m^{\mathrm{R}}$. We use factors based on portfolio values V_{it}, $i = 1, \dots, m$, where $\mathrm{d}V_{it} = V_{it}\mu_{it} \,\mathrm{d}t + V_{it} \sum_{j=1}^{m} \sigma_{ijt} \,\mathrm{d}W_{jt}^{\mathrm{R}}$, where $\mu_{it} = r_t + \sum_{j=1}^{m} \sigma_{ijt}\lambda_{jt}$. We consider stationary factors given by $X_{it} = e^{-\bar{\mu}_i t} V_{it}$, $i = 1, \dots, m$, where $\bar{\mu}_i$ are long-term growth rates, which satisfy

$$\mathrm{d}X_{it} = X_{it}(\mu_{it} - \bar{\mu}_j) \,\mathrm{d}t + X_{it} \sum_{j=1}^{m} \sigma_{ijt} \,\mathrm{d}W_t^{\mathrm{R}}. \tag{2}$$

In particular, we use $X_{Ct} = \sum_{i=1}^{m} X_{it}$ and specify the short rate, using an infinite-dimensional weight function $\psi(\tau)$, by

$$
\begin{aligned}
r_t &= \beta + \Xi_{Ct} - X_{Ct}, \qquad\qquad (3)\\
\Xi_{Ct} &= \int_{-\infty}^{t} \psi(t-s) G_s(t)\,\mathrm{d}s, \quad \text{where} \quad G_s(t) = \frac{\overline{P}(t-s)}{P_s(t)} X_{Cs},
\end{aligned}
$$

and $\overline{P}(\tau)$ is the long-term mean-reverting value of $P_t(T)$, where $T = t + \tau$. We divide X_{Cs} by the bond price, since the time t value of X_{is} is given by $X_{is}/P_s(t)$.

This specification can be used to derive arbitrage-free bond prices.

Theorem 1 *Based on the specification of the stochastic discount factor* (1), *the factors* (2) *and the short rate* (3), *the bond prices are given by*

$$
\begin{aligned}
P_t(T) &= e^{-\int_0^{\tau} \theta_t(u)\,\mathrm{d}u} \left(1 + \sum_{i=1}^{m} X_{it} h_i(\tau)\right),\\
h_i(\tau) &= \int_0^{\tau} \zeta_i(s) e^{\int_0^{s} \theta_t(u) - \bar{\mu}_i\,\mathrm{d}u}\,\mathrm{d}s, \quad i = 1, \ldots, m,\\
\theta_t(\tau) &= \beta + \int_{-\infty}^{t} \psi(t-s+\tau) G_s(t+\tau)\,\mathrm{d}s,\\
\zeta_i(\tau) &= 1 - \int_0^{\tau} \psi(s) \overline{P}(s) e^{\bar{\mu}_i s}\,\mathrm{d}s, \quad i = 1, \ldots, m.
\end{aligned}
$$

Proof Using (1) and (3), the bond prices are given by

$$
\begin{aligned}
P_t(T) &= \mathrm{E}_t(\Lambda_T/\Lambda_t) = 1 - \int_t^{T} \mathrm{E}_t(r_u \Lambda_u/\Lambda_t)\,\mathrm{d}u,\\
r_t &= \beta + \int_{-\infty}^{t} \psi(t-s) G_s(t)\,\mathrm{d}s - X_{Ct} = \beta + \Xi_t - \sum_{j=1}^{m} X_{jt}.
\end{aligned}
$$

Define

$$
\xi_t(\tau) = \theta_t(\tau) - \beta = \int_{-\infty}^{t} \psi(t+\tau-s) G_s(t)\,\mathrm{d}s,
$$

then $\Xi_t = \xi_t(0)$. As $\mathrm{E}_t(V_{jT}\Lambda_T/\Lambda_t) = V_{jt}$, we find, using (2), $X_{jt} = e^{\bar{\mu}_j(T-t)}\,\mathrm{E}_t(X_{jT}\Lambda_T/\Lambda_t)$.

Consequently,

$$P_t(T) = 1 - \beta \int_t^T P_t(u)\,\mathrm{d}u - \int_t^T \mathrm{E}_t\left\{\xi_u(0)\Lambda_u/\Lambda_t\right\}\mathrm{d}u + \sum_{j=1}^m \int_t^T e^{-\bar{\mu}_j(u-t)}\,\mathrm{d}u X_{jt},$$

and

$$\frac{\partial P_t(T)}{\partial T} = -\beta P_t(T) - \mathrm{E}_t\left\{\xi_T(0)\Lambda_T/\Lambda_t\right\} + \sum_{j=1}^m e^{-\bar{\mu}_j(T-t)} X_{jt}.$$

Let $\mathrm{E}_t\left\{\xi_T(0)\Lambda_T/\Lambda_t\right\} = c_1 + c_2$, where

$$c_1 = \mathrm{E}_t\left\{\left(\int_{-\infty}^t \psi(T-s)G_s(T)\,\mathrm{d}s\right)\Lambda_T/\Lambda_t\right\} = \xi_t(\tau)P_t(T),$$

and

$$\begin{aligned}
c_2 &= \mathrm{E}_t\left\{\left(\int_t^T \psi(T-s)G_s(T)\,\mathrm{d}s\right)\Lambda_T/\Lambda_t\right\} \\
&= \mathrm{E}_t\left\{\int_t^T \psi(T-s)\frac{\overline{P}(T-s)\sum_{j=1}^m X_{js}}{P_s(T)}\Lambda_s/\Lambda_t\,\mathrm{E}_s(\Lambda_T/\Lambda_s)\,\mathrm{d}s\right\} \\
&= \int_t^T \mathrm{E}_t\left\{(\psi(T-s)\overline{P}(T-s)\left\{\sum_{j=1}^m X_{js}\right\}\Lambda_s/\Lambda_t\right\}\mathrm{d}s \\
&= \sum_{j=1}^m X_{jt}\int_t^T \psi(T-s)\overline{P}(T-s)e^{-\bar{\mu}_j(s-t)}\,\mathrm{d}s \\
&= \sum_{j=1}^m X_{jt}e^{-\bar{\mu}_j(T-t)}\int_t^T \psi(T-s)\overline{P}(T-s)e^{\bar{\mu}_j(T-s)}\,\mathrm{d}s \\
&= \sum_{j=1}^m X_{jt}e^{-\bar{\mu}_j\tau}\int_0^\tau \psi(s)\overline{P}(s)e^{\bar{\mu}_j s}\,\mathrm{d}s.
\end{aligned}$$

We thus find

$$\begin{aligned}
\frac{\partial P_t(T)}{\partial T} &= -\left\{\beta + \xi_t(\tau)\right\}P_t(T) + \sum_j e^{-\bar{\mu}_j\tau}X_{jt}\left(1 - \int_0^\tau \psi(s)\overline{P}(s)e^{\bar{\mu}_j s}\,\mathrm{d}s\right) \\
&= -\theta_t(\tau)P_t(T) + \sum_j X_{jt}\zeta_j(\tau)e^{-\bar{\mu}_j\tau}, \qquad (4)
\end{aligned}$$

where $\theta_t(\tau) = \beta + \xi_t(\tau)$ and $\zeta_j(\tau) = 1 - \int_0^\tau \psi(s)\overline{P}(s)e^{\bar{\mu}_j s}\,\mathrm{d}s$.

From (4), which is a first-order, linear, inhomogeneous differential equation with function coefficients, we derive the solution

$$P_t(T) = e^{-\int_0^\tau \theta_t(u)\,\mathrm{d}u}\left(1 + \sum_j X_{jt}\int_0^\tau \zeta_j(s)e^{\int_0^s \theta_t(u)-\bar{\mu}_j\,\mathrm{d}u}\,\mathrm{d}s\right), \qquad \text{q.e.d.}$$

The model is non-Markov. The state vector formed by $X_{1t}, \ldots, X_{mt}$ and $\theta_t(\tau)$, $\tau \geq 0$, is infinite dimensional. However, we also find that all past information is contained in the present yield curve. Since the forward rates are given by

$$F_t(T) = \theta_t(\tau) - P_t^{-1}(T)\sum_i \zeta_i(\tau)e^{-\bar{\mu}_i\tau}X_{it},$$

the infinite dimensional state subvector $\theta_t(\tau)$ can be expressed as a function of the m present factors and $P_t(T)$. Current work aims at further exploring the properties of the model and applying it to US Treasury yields.

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