Summer Course Mathematics

Facts Wednesday

1. A function is a 'rule' that assigns a unique y-value to every x-value in the given domain.

   Example: $y = f(x) = x^2 - 5$. x is the independent variable, y is the dependent variable.

2. Domain of a function: Set of all values for the independent variable(s) for the function is defined.

   Range: Set of all possible outcomes of the function.

   If a function has name f, common notation for the domain and the range are D_f and R_f, respectively.

3. A function f is increasing if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$, with $x_1, x_2 \in D_f$.

   A function f is strictly increasing if $x_1 < x_2$ implies $f(x_1) < f(x_2)$, with $x_1, x_2 \in D_f$.

   A function f is decreasing if $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$, with $x_1, x_2 \in D_f$.

   A function f is strictly decreasing if $x_1 < x_2$ implies $f(x_1) > f(x_2)$, with $x_1, x_2 \in D_f$.

4. The graph of a function f shows the set of all points $(x, f(x))$, with x in the domain of f.

5. The general form of a linear function is

   $$y = f(x) = ax + b.$$

   The slope of a linear function is a, the intercept is b.

6. If two points (x_1, y_1) and (x_2, y_2) on the graph of a linear function are known, the slope of the linear function can be determined:

   $$a = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2}.$$

   After determining the slope a, the intercept b can also be determined using of the two given points.

7. Properties of a quadratic function $y = f(x) = ax^2 + bx + c$, $a \neq 0$, $x \in D_f$:

   - Three parameters: a, b, and c.
   - Two variables: x independent, y dependent.
   - The graph of a quadratic function has an axis of symmetry.
   - The graph of a quadratic function either has a minimum or a maximum.
   - If the coefficient a of x^2 is positive, the graph/function has a minimum.
   - If the coefficient a of x^2 is negative, graph/function has a maximum.

8. General power function:

   $$f(x) = Ax^r, \quad x > 0, \ r \in \mathbb{R}, \ A \in \mathbb{R}.$$

9. General exponential function:

$$f(t) = Aa^t, \qquad t \in \mathbb{R},\ a > 0,\ A \in \mathbb{R}.$$

Properties:

- If t increases by 1, $f(t)$ increases or decreases by the fixed factor a per unit of time.
- $f(0) = A$.
- If $a > 1$, then f is increasing.
- If $0 < a < 1$, then f is decreasing.
- If $a = 1$, then f is constant.
- If $a = 1 + \frac{p}{100}$, where $p > 0$ and $A > 0$, then $f(x)$ will increase by $p\,\%$ per unit of time.
- If $a = 1 - \frac{p}{100}$, where $0 < p < 100$ and $A > 0$, then $f(x)$ will decrease by $p\,\%$ per unit of time.

10. The doubling time for an exponential function is the time it takes for an increasing exponential function (i.e., $A > 0$ and $a > 0$) to double its value. It is the solution for t of $A^t = 2$.

11. The most important exponential function in mathematics is the exponential function with base e:

$$f(x) = \mathrm{e}^x.$$

This function is called the natural exponential function. The number e is an irrational number. It is approximately equal to

$$\mathrm{e} = 2.718281828459045\ldots$$

12. Rules for powers obviously also apply to the natural exponential function:

- $\mathrm{e}^s\mathrm{e}^t = \mathrm{e}^{s+t}$.
- $\dfrac{\mathrm{e}^s}{\mathrm{e}^t} = \mathrm{e}^{s-t}$.
- $(\mathrm{e}^s)^t = \mathrm{e}^{st}$.

13. If $\mathrm{e}^u = a$, we call u the natural logarithm of a, and we write $u = \ln a$.

14. Properties of the natural logarithmic function ln:

- $\ln(xy) = \ln x + \ln y, \qquad x, y > 0.$
- $\ln \frac{x}{y} = \ln x - \ln y, \qquad x, y > 0.$
- $\ln x^p = p \ln x, \qquad x > 0.$
- $\ln 1 = 0.$
- $\ln \mathrm{e} = 1.$
- $x = \mathrm{e}^{\ln x}, \qquad x > 0.$
- $\ln \mathrm{e}^x = x.$

Note that $\ln(x + y)$ and $\ln(x - y)$ cannot be rewritten.

15. If $a^u = x$, $a > 0$, we call u the logarithm of x to base a, and we write $u = \log_a x$.

Note that $\log_{\mathrm{e}}$ is equivalent to ln.

16. A logarithm to a certain base a can be rewritten in terms of a logarithm to any other base b:

$$\log_a x = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}.$$

17. Equations with exponential functions can be solved by taking the natural logarithm on both sides and applying the rules for the natural logarithmic function.

    Example: $2^x = 31 \Rightarrow \ln 2^x = \ln 31 \Rightarrow x \ln 2 = \ln 31 \Rightarrow x = \frac{ln 31}{\ln 2} \approx 4.95$.

18. If an equation contains an exponential function, and if we can write both sides with the same base, we don't need logarithms to solve the equation.

    Example: $2^x = 32 \Rightarrow 2^x = 2^5 \Rightarrow x = 5$.