Summer Course Mathematics

Exercises Friday

Chapter 6: Differentiation

§ 8 The Chain Rule

1. Use the chain rule to determine $\mathrm{d}y/\mathrm{d}x$ for the following functions:
   a. $y = (x^3+5)^7$
   b. $y = (3x^2+4x+5)^2 + \dfrac{1}{3x^2+4x+5}$
   c. $y = 6(x^3+\frac{2}{x})^2 + 5(x^3+\frac{2}{x}) + 7$
2. Determine the derivatives of the following functions:
   a. $y = f(x) = \left(\dfrac{x+1}{x-1}\right)^7$
   b. $y = g(u) = 3u^2\left(\dfrac{1}{u}+u^3\right)^4$
3. Assume that f is a differentiable function. Determine expressions for the derivatives of the following functions:
   a. $x^2 + f(x)$ b. $(f(x))^2 - \frac{3}{x}$ c. $(x+f(x))^3$
   d. $x^2(f(x))^3$ **e.** $\sqrt{xf(x)}$ f. $\dfrac{f(x)+1}{f(x)-1}$
4. Use the chain rule to determine $\mathrm{d}y/\mathrm{d}x$ for the following functions.
   a. $y = 8(8-x^3)^3$
   b. $y = 2\sqrt{1-\frac{1}{x}}$

§ 10 Exponential Functions

1. Determine the derivatives of the following functions:
   a. $y = 2\mathrm{e}^x + 3$ b. $y = \dfrac{3\mathrm{e}}{\mathrm{e}^x}$ c. $y = 4\mathrm{e}^{-2x} + x^2 + 4$
   d. $y = \dfrac{\mathrm{e}^x}{\mathrm{e}^{3x}}$ e. $y = (x^2+1)\mathrm{e}^{3x}$ **f.** $y = (\mathrm{e}^x+3)^5$
   g. $y = \mathrm{e}^{x^2+4x+3}$

Chapter 9: Optimization

§ 1 Extreme Points

1. Use non-calculus arguments to find the maximum or minimum points for the following functions:

   a. $y = f(x) = (x-1)^2 + 4$

   b. $y = g(x) = 4 - (x-1)^2$

   c. $y = h(x) = \dfrac{12}{(x-2)^2 + 2}$

   d. $y = k(x) = \dfrac{6}{2 - (x-2)^2}$

2. Use non-calculus arguments to find the maximum or minimum points for the following functions:

   a. $y = l(x) = 4 + \mathrm{e}^{1-x^2}$

   b. $y = m(x) = 1 + \ln(4 + x^2)$

   c. $y = n(x) = 1 + \sqrt{1 + x^2}$

   d. $y = p(x) = \dfrac{4}{1 + \sqrt{x-1}}$

§ 2 Simple Tests for Extreme Points

1. Describe in your own words the relation between f and f', if there is any, with respect to:

   a. Sign of f and the behaviour of f'.

   b. Sign of f' and the behaviour of f.

   c. Zero of f and the behaviour of f'.

   d. Zero of f' and the behaviour of f.

   e. Explain why a sign diagram is useful in this setting.

2. The function f is defined as $f(x) = \dfrac{x}{\mathrm{e}^x}$.

   a. Determine the derivative $f'(x)$ of $f(x)$.

   b. Determine the stationary points of f.

   c. Determine the intervals where f increases and decreases.

   d. Determine the extreme points (maximum and/or minimum points) of f.

3. Let $f(x) = \dfrac{x^2}{x^2+5}$. Determine $f'(x)$ and use its sign variation to determine where $f(x)$ is increasing and where it is decreasing.

4. A firm's production function is $Q(L) = 24L^2 - \frac{1}{10}L^3$, where L denotes the number of workers with $0 \leq L \leq 200$.

   a. What size of the work force (let us denote this by L^*) maximizes the output $Q(L)$?

   b. What size of the work force (let us denote this by L^{**}) maximizes the output per worker $Q(L)/L$?

§ 6 Local Extreme Points

1. Determine possible local extreme points (maxima and minima) for the following functions.

   a. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$.

   b. $g(x) = 2x^2 - 8x + 7$.

   c. $h(x) = x^3 - 3x + 9$.

   d. $p(x) = x - \frac{4}{x^2}$.

   e. $q(x) = x\mathrm{e}^{2x}$.

   f. $r(x) = 3x^4 - 12x^3 - 24x^2 + 12$.

2. Let $f(x) = \frac{x}{x^2+4}$.

   a. Determine the derivative $f'(x)$ of the function $f(x)$.

   b. Factorize $f'(x)$ and use a sign diagram to determine its sign variation.

   c. Determine the stationary points of the function $f(x)$ and determine where it is increasing/decreasing.

   d. Use your answer to c. to identify and classify the extreme points $f(x)$.

3. Let $f(x) = x^2\mathrm{e}^{\frac{1}{2}x^2+3x}$.

   a. Determine the derivative $f'(x)$ of the function $f(x)$.

   b. Factorize $f'(x)$ and use a sign diagram to determine its sign variation.

   c. Determine the stationary points of the function $f(x)$ and determine where it is increasing/decreasing.

   d. Use your answer to c. to identify and classify the extreme points of $f(x)$.

Continue with the following problems in the book:

- Chapter 6, § 8: Problems 1, 10.
- Chapter 6, § 10: Problem 4.