Answers Sample Entrance Exam Mathematics

Duration: 3 hours

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1. Basics, I.

   a. (3) $x = 2$.

   b. (4)
   (i) $2 < -4$ is not true, so $x = 0$ is not a solution.
   (ii) $-x + 2 < 2x - 4 \Rightarrow 6 < 3x \Rightarrow x > 2$.

   c. (4) $x^2 - 3x + 2 = 0 \Rightarrow (x-1)(x-2) = 0 \Rightarrow x - 1 = 0$ or $x - 2 = 0 \Rightarrow x = 1$ or $x = 2$.

   d. (5) $x = 2, y = 1$.

   e. (4)
   (i) $4x^3 - 100x = 4x(x^2 - 25) = 4x(x+5)(x-5)$.
   (ii) $x^2 + 5x + 6 = (x+2)(x+3)$.

   f. (3)
   (i) $16x^2 = 64 \Rightarrow x^2 = 4 \Rightarrow x = 2$ or $x = -2$.
   (ii) $27^{2x-2} = 81^x \Rightarrow 3^{3(2x-2)} = 3^{4x} \Rightarrow 6x - 6 = 4x \Rightarrow 2x = 6 \Rightarrow x = 3$.
   (iii) $\ln(x) + \ln(2x) = \ln(8), x > 0 \Rightarrow 2x^2 = 8, x > 0 \Rightarrow x = 2$.

   g. (3)
   (i) $x = 0.63$.
   (ii) $7^{3x+1} = 98 \Rightarrow 3x + 1 = \frac{\ln 98}{\ln 7} \Rightarrow x = 0.45$.
   (iii) $\log_3 x = 5.5 \Rightarrow x = 3^{5.5} = 420.89$.

2. Basics, II.

   a. (4) $\dfrac{2}{x+3} + \dfrac{7}{x+2} = -1, x \neq -3, x \neq -2$
   $\Rightarrow 2(x+2) + 7(x+3) + (x+2)(x+3) = 0, x \neq -3, x \neq -2$
   $\Rightarrow x^2 + 14x + 31 = 0, x \neq -3, x \neq -2$
   $\Rightarrow x = -11.24$ or $x = -2.76$.

   b. (4) $(x-2)\sqrt{x-1} = 0 \Rightarrow x - 2 = 0$ or $x - 1 = 0, x \geq 1 \Rightarrow x = 2$ or $x = 1$.

   c. (4) $\ln(\frac{1}{3}x^{-2}) = \ln(\frac{1}{3}) + \ln(x^{-2}) = \ln 1 - \ln 3 - 2\ln x = -\ln 3 - 2\ln x,\ x > 0$.

   d. (5) $y = ax + b, a = \frac{19-9}{14-9} = 2 \Rightarrow 9 = 2 \cdot 9 + b \Rightarrow b = -9 \Rightarrow y = 2x - 9$.

   e. (6) $\dfrac{x-4}{x-1} \geq 2 \Rightarrow \dfrac{x-4}{x-1} - 2 \geq 0 \Rightarrow \dfrac{x-4}{x-1} - \dfrac{2(x-1)}{x-1} \geq 0 \Rightarrow \dfrac{x-4-2x+2}{x-1} \geq 0 \Rightarrow \dfrac{-x-2}{x-1} \geq 0 \Rightarrow \dfrac{x+2}{x-1} \leq 0$. Based on a sign diagram it follows that the solution is $-2 \leq x < 1$.

3. Differentiation and shifting graphs.

   a. (4) $f'(x) = \frac{1}{2\sqrt{x}} + 4x^3, x \geq 0$.
   $f'(1) = 4\frac{1}{2} > 0 \Rightarrow f$ is increasing at $x = 1$.

   b. (4) $g'(x) = (3x^2 + 2)\mathrm{e}^{2x+1} + (x^3 + 2x + 1) \cdot 2\mathrm{e}^{2x+1}$.

   c. (4) $h'(x) = 3(x^4 + 4x^2 + 1)^2 \cdot (4x^3 + 8x)$.
   $h'(0) = 3(1)^2 \cdot 0 = 0 \Rightarrow h$ is neither increasing or decreasing, but stationary.

   d. (3) Shift 3 units to the left, stretch the graph in the positive Y-direction with a factor 3, shift 2 units upwards.

4. Growth processes.

   a. (2) $15000(1.024)^{-10} = 11832.91$.

   b. (2) $15000(1.0255)^5 = 17012.56$.

   c. (2) $25500(0.88)^5 = 13457.16$.

5. Extremes.

   a. (4) f is negative for $x < -3$, f is positive for $-3 < x < 0$, and f is negative for $0 < x < 3$, thus f has a maximum somewhere in the interval from -3 to 0.
   f is positive for $-3 < x < 0$, f is negative for $0 < x < 3$, and f is positive for $x > 3$, thus f has a minimum somewhere in the interval from 0 to 3.

   b. (2) $x(x-3)(x+3) = x(x^2-9) = x^3 - 9x$.

   c. (4) $f'(x) = 3x^2 - 9$.
   $f'(x) = 3x^2 - 9 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3} \Rightarrow x = \pm 1.7$.

   d. (4) Make a sign diagram of $f'(x) = 3(x^2-3) = 3(x-\sqrt{3})(x+\sqrt{3})$. It follows that f is increasing for $x \leq -\sqrt{3}$ and for $x \geq \sqrt{3}$ and that f is decreasing for $-\sqrt{3} \leq x \leq \sqrt{3}$.

   e. (3) At $x = -\sqrt{3}$, f goes from increasing to decreasing. Thus, f has a maximum at $x = -\sqrt{3}$.
   At $x = \sqrt{3}$, f goes from decreasing to increasing. Thus, f has a minimum at $x = \sqrt{3}$.

   f. (3) Use the points $(-\sqrt{3}, 10.4)$ and $(\sqrt{3}, -10.4)$.

6. **For aspirant students Econometrics and Operations Research only!**

   a. (2)
   (i) $\sin(212^o) \approx -0.53$.
   (ii) $\cos(\frac{1}{3}\pi) = 0.5$.

   b. (4) The range of sin is $[-1, 1]$, so the range of $5 - 3\sin(t-2)$ is $[2, 8]$. The equation does not have a solution because 0 is not contained in this range.
   $5 - 6\sin(t-2) = 0 \Rightarrow \frac{5}{6} = \sin(t-2) \Rightarrow$ possibility: $t - 2 \approx 0.9851 \Rightarrow t \approx 2.9852$.

   c. (3) We can only take the square root of nonnegative numbers $\Rightarrow x + 2 \geq 0 \Rightarrow x \geq -2$. The outcome of a square root is always nonnegative $\Rightarrow 1 - x \leq 0 \Rightarrow x \geq 1$. By combining these two conditions, it follows that the domain is $x \geq 1$.
   $-2\sqrt{x+2} = 1 - x \Rightarrow 4(x+2) = (1-x)^2 \Rightarrow x^2 - 6x - 7 = 0 \Rightarrow (x-7)(x+1) = 0 \Rightarrow$ $x = 7$ or $x = -1$. Only $x = 7$ falls within the domain $x \geq 1$.
   Conclusion: The unique solution of $-2\sqrt{x+2} = 1 - x$ is $x = 7$.

   d. (4) Conditions: $x \neq -1$, $x \neq 1$. Solution: $x = 2$ ($x = 1$ does not satisfy the conditions).

   e. (4) $f(x) = x\mathrm{e}^x \Rightarrow f'(x) = (1+x)\mathrm{e}^x \Rightarrow f''(x) = (2+x)\mathrm{e}^x$.
   Stationary point: $x = -1$. $f''(-1) = \mathrm{e}^{-1} > 0 \Rightarrow$ The stationary point is a minimum.

   f. (4)
   (i) $\lim\limits_{x\to 1} \dfrac{x-1}{x^2-1} = \lim\limits_{x\to 1} \dfrac{x-1}{(x+1)(x-1)} = \lim\limits_{x\to 1} \dfrac{1}{x+1} = \frac{1}{2}$.
   (ii) $\lim\limits_{x\to -\infty} \dfrac{x|x|-2}{x^2+2} = \lim\limits_{x\to -\infty} \dfrac{-x^2-2}{x^2+2} = \lim\limits_{x\to -\infty} -1 = -1$.

   g. (4)
   (i) $\displaystyle\int (6x^2 + 5)\ \mathrm{d}x = 2x^3 + 5x + C$.
   (ii) $\displaystyle\int_0^2 (6x^2 + \sqrt{x})\ \mathrm{d}x = \Big|_0^2 (2x^3 + \tfrac{2}{3}x^{1\frac{1}{2}}) = 2 \cdot 2^3 + \tfrac{2}{3} \cdot 2^{1\frac{1}{2}} - 0 \approx 17.89$.